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Emmanuel Fricain, Andreas Hartmann

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# REGULARITY ON THE BOUNDARY IN SPACES OF HOLOMORPHIC FUNCTIONS ON THE UNIT DISK

EMMANUEL FRICAIN & ANDREAS HARTMANN

**ABSTRACT.** We review some results on regularity on the boundary in spaces of analytic functions on the unit disk connected with backward shift invariant subspaces in  $H^p$ .

## 1. INTRODUCTION

Fatou's theorem shows that every function of the Nevanlinna class  $N := \{f \in \text{Hol}(\mathbb{D}) : \sup_{0 < r < 1} \int_{-\pi}^{\pi} \log_+ |f(re^{it})| dt < \infty\}$  admits non tangential limits at almost every point  $\zeta$  of the unit circle  $\mathbb{T} = \partial\mathbb{D}$ ,  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  being the unit disk. One can easily construct functions (even contained in smaller classes) which do not admit non-tangential limits on a dense set of  $\mathbb{T}$ . The question that arises from such an observation is whether one can gain regularity of the functions at the boundary when restricting the problem to interesting subclasses of  $N$ . We will discuss two kinds of subclasses corresponding to two different ways of generalizing the class of standard backward shift invariant subspaces in  $H^2 := \{f \in \text{Hol}(\mathbb{D}) : \|f\|_2^2 := \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^2 dt < \infty\}$ . Recall that backward shift invariant subspaces have shown to be of great interest in many domains in complex analysis and operator theory. In  $H^2$ , they are given by  $K_I^2 := H^2 \ominus IH^2$ , where  $I$  is an inner function, that is a bounded analytic function in  $\mathbb{D}$  the boundary values of which are in modulus equal to 1 a.e. on  $\mathbb{T}$ . Another way of writing  $K_I^2$  is

$$K_I^2 = H^2 \cap I\overline{H_0^2},$$

where  $H_0^2 = zH^2$  is the subspace of functions in  $H^2$  vanishing in 0. The bar sign means complex conjugation here. This second writing  $K_I^2 = H^2 \cap I\overline{H_0^2}$  does not appeal to the Hilbert space structure and thus generalizes to  $H^p$  (which is defined as  $H^2$  but replacing the integration power 2 by  $p \in (0, \infty)$ ; it should be noted that for  $p \in (0, 1)$  the expression  $\|f\|_p^p$  defines a metric; for  $p = \infty$ ,  $H^\infty$  is the Banach space of bounded analytic functions on  $\mathbb{D}$  with obvious norm). When  $p = 2$ , then these spaces are also called model spaces because they arise in the construction of a universal model for Hilbert space contractions developed by Sz.-Nagy–Foias (see [SNF67]). Note that if  $I$  is a Blaschke product associated with a sequence  $(z_n)_{n \geq 1}$  of points in  $\mathbb{D}$ , then  $K_I^p$  coincides with the closed linear span of simple fractions with the poles of corresponding multiplicities at the points  $1/\overline{z_n}$ .

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Many questions concerning regularity on the boundary for functions in standard backward shift invariant subspaces were investigated in the extensive existing literature. In particular, it is natural to ask whether one can find points in the boundary where every function  $f$  in  $K_I^p$  and its derivatives up to a given order have non tangential limits; or even can one find some arc on the boundary where every function  $f$  in  $K_I^p$  can be continued analytically? Those questions were investigated by Ahern–Clark, Cohn, Moeller,... Another interest in backward shift invariant subspaces concerns embedding questions, especially when  $K_I^p$  embeds into some  $L^p(\mu)$ . This question is related to the famous Carleson embedding theorem and was investigated for instance by Aleksandrov, Cohn, Treil, Volberg and many others (see below for some results).

In this survey, we will first review the important results in connection with regularity questions in standard backward shift invariant subspaces. Then we will discuss these matters in the two generalizations we are interested in: de Branges–Rovnyak spaces on the one hand, and weighted backward shift invariant subspaces — which occur naturally in the context of kernels of Toeplitz operators — on the other hand. Results surveyed here are mainly not followed by proofs. However, some of the material presented in Section 4 is new. In particular Theorem 18 for which we provide a proof and Example 4.1 that we will discuss in more detail. The reader will notice that for the de Branges–Rovnyak situation there now exists a quite complete picture analogous to that in the standard  $K_I^p$  spaces whereas the weighted situation has not been investigated very much yet. The example 4.1 should convince the reader that the weighted situation is more intricate in that the Ahern–Clark condition even under strong conditions on the weight — that ensure e.g. analytic continuation off the spectrum of the inner function — is not sufficient.

## 2. BACKWARD SHIFT INVARIANT SUBSPACES

We will need some notation. Recall that the spectrum of an inner function  $I$  is defined as  $\sigma(I) = \{\zeta \in \text{clos } \mathbb{D} : \liminf_{z \rightarrow \zeta} |I(z)| = 0\}$ . This set corresponds to the zeros in  $\mathbb{D}$  and their accumulation points on  $\mathbb{T} = \partial\mathbb{D}$ , as well as the closed support of the singular measure  $\mu_S$  of the singular factor of  $I$ .

The first important result goes back to Moeller [Mo62] (see also [AC69] for a several variable version):

**Theorem 1** (Moeller, 1962). *Let  $\Gamma$  be an open arc of  $\mathbb{T}$ . Then every function  $f \in K_I^p$  can be continued analytically through  $\Gamma$  if and only if  $\Gamma \cap \sigma(I) = \emptyset$ .*

Moeller also establishes a link with the spectrum of the compression of the backward shift operator to  $K_I^p$ .

It is of course easy to construct inner functions the spectrum of which on  $\mathbb{T}$  is equal to  $\mathbb{T}$  so that there is no analytic continuation possible. Take for instance for  $I$  the Blaschke product associated with the sequence  $\Lambda = \{(1 - 1/n^2)e^{in}\}_n$ , the zeros of which accumulate at every point on  $\mathbb{T}$ . So it is natural to ask what happens in points which are in the spectrum, and what kind of regularity can be expected there. Ahern–Clark and Cohn gave an answer to this question in [AC70, Co86a]. Recall that an arbitrary inner function  $I$  can be factored into a Blaschke product and a singular

inner function:  $I = BS$ , where  $B = \prod_n b_{a_n}$ ,  $b_{a_n}(z) = \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z}$ ,  $\sum_n (1 - |a_n|^2) < \infty$ , and

$$S(z) = \exp \left( - \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu_S(\zeta) \right),$$

where  $\mu_S$  is a finite positive measure on  $\mathbb{T}$  singular with respect to Lebesgue measure. The regularity of functions in  $K_I^2$  is then related with the zero distribution of  $B$  and the measure  $\mu_S$  as indicated in the following result.

**Theorem 2** (Ahern–Clark, 1970, Cohn, 1986). *Let  $I$  be an inner function and let  $1 < p < +\infty$  and  $q$  its conjugated exponent. If  $N$  is a non-negative integer and  $\zeta \in \mathbb{T}$ , then the following are equivalent:*

- (i) *for every  $f$  in  $K_I^p$ , the functions  $f^{(j)}$ ,  $0 \leq j \leq N$ , have finite non-tangential limits at  $\zeta$ ;*
- (ii) *the following condition holds:*

$$(1) \quad S_{q,N}^I(\zeta) := \sum_{n=1}^{\infty} \frac{(1 - |a_n|^2)}{|1 - \bar{\zeta} a_n|^{(N+1)q}} + \int_0^{2\pi} \frac{1}{|1 - \bar{\zeta} e^{i\theta}|^{(N+1)q}} d\mu_S(\theta) < +\infty.$$

Moreover in that case, the function  $(k_\zeta^I)^{N+1}$  belongs to  $K_I^q$  and we have

$$(2) \quad f^{(N)}(\zeta) = N! \int_{\mathbb{T}} \bar{z}^N f(z) \overline{k_\zeta^I(z)}^{N+1} dm(z),$$

for every function  $f \in K_I^p$ .

Here  $k_\zeta^I$  is the reproducing kernel of the space  $K_I^2$  corresponding to the point  $\zeta$  and defined by

$$(3) \quad k_\zeta^I(z) = \frac{1 - \overline{I(\zeta)} I(z)}{1 - \bar{\zeta} z}.$$

The case  $p = 2$  is due to Ahern–Clark and Cohn generalizes the result to  $p > 1$ . Another way to read into the results of Ahern–Clark, Cohn and Moeller is to introduce the representing measure of the inner function  $I$ ,  $\mu_I = \mu_S + \mu_B$ , where

$$\mu_B := \sum_{n \geq 1} (1 - |a_n|^2) \delta_{\{a_n\}}.$$

Then Theorems 1 and 2 allow us to formulate the following general principle: if the measure  $\mu_I$  is “small” near a point  $\zeta \in \mathbb{T}$ , then the functions  $f$  in  $K_I^p$  must be smooth near that point.

Another type of regularity questions in backward shift invariant subspaces was studied by A. Aleksandrov, K. Dyakonov and D. Khavinson. It consists in asking if  $K_I^p$  contains a nontrivial smooth function. More precisely, Aleksandrov in [Al81] proved that the set of functions  $f \in K_I^p$  continuous in the closed unit disc is dense in  $K_I^p$ . It should be noted nevertheless that the result of Aleksandrov is not constructive and indeed we do not know how to construct explicit examples of functions  $f \in K_I^p$  continuous in the closed unit disc. In the same direction, Dyakonov and Khavinson, generalizing a result by Shapiro on the existence of  $\mathcal{C}^n$ -functions in  $K_I^p$  [Sh67], proved in [DK06] that the space  $K_I^2$  contains a nontrivial function of class  $\mathcal{A}^\infty$  if and only if either  $I$  has a zero in  $\mathbb{D}$  or there is a Carleson set  $E \subset \mathbb{T}$  with  $\mu_S(E) > 0$ ; here  $\mathcal{A}^\infty$  denotes the space of analytic functions on  $\mathbb{D}$  that extend continuously to the closed unit disc and that are

$C^\infty(\mathbb{T})$ ; recall that a set  $E$  included in  $\mathbb{T}$  is said to be a Carleson set if the following condition holds

$$\int_{\mathbb{T}} \log \text{dist}(\zeta, E) dm(\zeta) > -\infty.$$

In [Dy08a, Dy02b, Dy00, Dy91], Dyakonov studied some norm inequalities in coinvariant backward shift subspaces of  $H^p(\mathbb{C}_+)$ ; here  $H^p(\mathbb{C}_+)$  is the Hardy space of the upper half-plane  $\mathbb{C}_+ := \{z \in \mathbb{C} : \text{Im } z > 0\}$  and if  $\Theta$  is an inner function for the upper half-plane, then the corresponding backward shift invariant subspace of  $H^p(\mathbb{C}_+)$  is also denoted by  $K_\Theta^p$  and defined to be

$$K_\Theta^p = H^p(\mathbb{C}_+) \cap \overline{\Theta H^p(\mathbb{C}_+)}.$$

In the special case where  $\Theta(z) = e^{iaz}$  ( $a > 0$ ), the space  $K_\Theta^p$  is equal to  $PW_a^p \cap H^p(\mathbb{C}_+)$ , where  $PW_a^p$  is the Paley-Wiener space of entire functions of exponential type at most  $a$  that belong to  $L^p$  on the real axis. Dyakonov shows that several classical regularity inequalities pertaining to  $PW_a^p$  apply also to  $K_\Theta^p$  provided  $\Theta'$  is in  $H^\infty(\mathbb{C}_+)$  (and only in that case). In particular, he proved the following result.

**Theorem 3** (Dyakonov, 2000 & 2002). *Let  $1 < p < +\infty$  and let  $\Theta$  be an inner function in  $H^\infty(\mathbb{C}_+)$ . The following are equivalent:*

- (i)  $K_\Theta^p \subset C_0(\mathbb{R})$ .
  - (ii)  $K_\Theta^p \subset L^q(\mathbb{R})$ , for some (or all)  $q \in (p, +\infty)$ .
  - (iii) The differentiation operator is bounded as an operator from  $K_\Theta^p$  to  $L^p(\mathbb{R})$ , that is
- $$(4) \quad \|f'\|_p \leq C(p, \Theta) \|f\|_p, \quad f \in K_\Theta^p.$$
- (iv)  $\Theta' \in H^\infty(\mathbb{C}_+)$ .

Notice that in (4) one can take  $C(p, \Theta) = C_1(p) \|\Theta'\|_\infty$ , where  $C_1(p)$  depends only on  $p$  but not on  $\Theta$ . Moreover, Dyakonov also showed that the embeddings in (i), (ii) and the differentiation operator on  $K_\Theta^p$  are compact if and only if  $\Theta$  satisfies (iv) and  $\Theta'(x) \rightarrow 0$  as  $|x| \rightarrow +\infty$  on the real line. In [Dy02a], the author discusses when the differentiation operator is in Schatten-von Neumann ideals. Finally in [Dy08a], Dyakonov studied coupled with (4) the reverse inequality. More precisely, he characterized those  $\Theta$  for which the differentiation operator  $f \mapsto f'$  provides an isomorphism between  $K_\Theta^p$  and a closed subspace of  $H^p$ , with  $1 < p < +\infty$ ; namely he showed that such  $\Theta$ 's are precisely the Blaschke products whose zero-set lies in some horizontal strip  $\{a < \text{Im } z < b\}$ , with  $0 < a < b < +\infty$  and splits into finitely many separated sequences.

The inequality (4) corresponds for the case  $\Theta(z) = e^{iaz}$  to a well-known inequality of S. Bernstein (see [Bern26, Premier lemme, p.75] or [Le96, Lecture 28]). For  $p = +\infty$ , a beautiful generalization of Bernstein's inequality was obtained by Levin: let  $x \in \mathbb{R}$  and  $|\Theta'(x)| < +\infty$ ; then for each  $f \in K_\Theta^\infty$ , the derivative  $f'(x)$  exists in the sense of non-tangential boundary values and

$$\left| \frac{f'(x)}{\Theta'(x)} \right| \leq \|f\|_\infty, \quad f \in K_\Theta^\infty.$$

Recently, differentiation in the backward shift invariant subspaces  $K_\Theta^p$  was studied extensively by A. Baranov. In [Ba03, Ba05b], he proved for a general inner function  $\Theta$  estimates of the form

$$(5) \quad \|f^{(n)} \omega_{p,n}\|_{L^p(\mu)} \leq C \|f\|_p, \quad f \in K_\Theta^p,$$

where  $n \geq 1$ ,  $\mu$  is a Carleson measure in the closed upper half-plane and  $\omega_{p,n}$  is some weight related to the norm of reproducing kernels of the space  $K_\Theta^2$  which compensates for possible growth of the derivative near the boundary. More precisely, put

$$\omega_{p,n}(z) = \|(k_z^\Theta)^{n+1}\|_q^{-\frac{p}{p+1}}, \quad (z \in \text{clos}(\mathbb{C}_+)),$$

where  $q$  is the conjugate exponent of  $p \in [1, +\infty)$ . We assume that  $\omega_{p,n}(x) = 0$ , whenever  $S_{q,N}^\Theta(x) = +\infty$ ,  $x \in \mathbb{R}$  (here we omit the exact formula of  $k_z^\Theta$  and  $S_{N,q}^\Theta$  in the upper half-plane but it is not difficult to imagine what will be the analogue of (1) and (3) in that case).

**Theorem 4** (Baranov, 2005). *Let  $\mu$  be a Carleson measure in  $\text{clos}(\mathbb{C}_+)$ ,  $1 \leq p < +\infty$ . Then the operator*

$$(T_{p,n}f)(z) = f^{(n)}(z)\omega_{p,n}(z)$$

*is of weak type  $(p, p)$  as an operator from  $K_\Theta^p$  to  $L^p(\mu)$  and is bounded as an operator from  $K_\Theta^r$  to  $L^r(\mu)$  for any  $r > p$ ; moreover there is a constant  $C = C(\mu, p, r, n)$  such that*

$$\|f^{(n)}\omega_{p,n}\|_{L^r(\mu)} \leq C\|f\|_r, \quad f \in K_\Theta^r.$$

To apply Theorem 4, one should have effective estimates of the considered weights, that is, of the norms of reproducing kernels. Let

$$\Omega(\Theta, \varepsilon) := \{z \in \mathbb{C}_+ : |\Theta(z)| < \varepsilon\}$$

be the level sets of the inner function  $\Theta$  and let  $d_\varepsilon(x) = \text{dist}(x, \Omega(\Theta, \varepsilon))$ ,  $x \in \mathbb{R}$ . Then Baranov showed in [Ba05b] the following estimates

$$(6) \quad d_\varepsilon^n(x) \lesssim \omega_{p,n}(x) \lesssim |\Theta'(x)|^{-n}, \quad x \in \mathbb{R}.$$

In fact, the inequality (6) is proved in [Ba05b, Lemma 4.5] for  $n = 1$ ; but the argument extends to general  $n$  in an obvious way. In [Al99], A. Aleksandrov proved that for the special class of inner functions  $\Theta \in CLS$  (see below for the definition), then we have

$$\omega_{p,n}(x) \asymp |\Theta'(x)|^{-n} \quad (x \in \mathbb{R}).$$

We should mention that Theorem 4 implies Theorem 3 on boundeness of differentiation operator. Indeed if  $\Theta' \in L^\infty(\mathbb{R})$ , then it is clear (and well known) that  $\sup_{x \in \mathbb{R}} \|k_x^2\|_q < +\infty$ , for any  $q \in (1, \infty)$ . Thus the weights  $\omega_r = \omega_{r,1}$  are bounded from below and thus inequality

$$\|f'\omega_r\|_p \leq C\|f\|_p \quad (f \in K_\Theta^p)$$

implies inequality (4). The proof of Baranov's result is based on the integral representation (2) which reduces the study of differentiation operators to the study of certain integral singular operators.

Another type of results concerning regularity on the boundary for functions in standard backward shift invariant subspaces is related to Carleson's embedding theorem. Recall that Carleson proved (see [Ca58] and [Ca62]) that  $H^p(\mathbb{C}_+)$  embeds continuously in  $L^p(\mu)$  (where  $\mu$  is a Borel measure on  $\text{clos}(\mathbb{C}_+)$ ) if and only if  $\mu$  is a Carleson measure, that is there is a constant  $C = C(\mu) > 0$  such that

$$\mu(S(x, h)) \leq Ch,$$

for every square  $S(x, h) = [x, x + h] \times [0, h] = \{t + iu : x \leq t \leq x + h, 0 \leq u \leq h\}$ ,  $x \in \mathbb{R}$ ,  $h > 0$ . The motivation of Carleson comes from interpolation problems but his result acquired wide importance in a larger context of singular integrals of Calderon–Zygmund type. In [Co82], Cohn studied a similar question for model subspaces  $K_\Theta^2$ . More precisely, he asked the following question: given an inner function  $\Theta$  in  $\mathbb{C}_+$  and  $p \geq 1$ , can we describe the class of Borel measure  $\mu$  in the closed upper half-plane such that  $K_\Theta^p$  is embedded into  $L^p(\mu)$ ? In spite of a number of beautiful and deep (partial) results, this problem is still open. Of course, due to the closed graph theorem, the embedding  $K_\Theta^p \subset L^p(\mu)$  is equivalent to the estimate

$$(7) \quad \|f\|_{L^p(\mu)} \leq C \|f\|_p \quad (f \in K_\Theta^p).$$

Cohn solved this question for a special class of inner functions. We recall that  $\Theta$  is said to satisfy the connected level set condition (and we write  $\Theta \in CLS$ ) if the set  $\Omega(\Theta, \varepsilon)$  is connected for some  $\varepsilon \in (0, 1)$ .

**Theorem 5** (Cohn, 1982). *Let  $\mu$  be a Borel measure on  $\text{clos}(\mathbb{C}_+)$ . Let  $\Theta$  be an inner function such that  $\Theta \in CLS$ . The following are equivalent:*

- (i)  $K_\Theta^2$  embeds continuously in  $L^2(\mu)$ .
- (ii) There is  $c > 0$  such that

$$(8) \quad \int_{\text{clos}(\mathbb{C}_+)} \frac{\text{Im } z}{|\zeta - \bar{z}|^2} d\mu(\zeta) \leq \frac{C}{1 - |\Theta(z)|}, \quad z \in \mathbb{C}_+.$$

The Theorem of Cohn can be reformulated in the following: the inequality (8) means exactly that the inequality 7 is satisfied for the reproducing kernels of the space  $K_\Theta^2$ . Thus to get inequality 7, it is sufficient to show it on reproducing kernels. Recently, F. Nazarov and A. Volberg [NV02] showed that this is no longer true in the general case. We should compare this property of the embedding operator  $K_\Theta^p \subset L^p(\mu)$  (for CLS inner functions) to the ”reproducing kernel thesis”, which is shared by Toeplitz or Hankel operators in  $H^2$  for instance. The reproducing kernel thesis says roughly that in order to show the boundeness of an operator on a reproducing kernel Hilbert space, it is sufficient to test its boundeness only on reproducing kernels (see e.g. [Ni02, Vol 1, p.131, 204, 244, 246] for some discussions of this remarkable property).

A geometric condition on  $\mu$  sufficient for the embedding of  $K_\Theta^p$  is due to Volberg–Treil [TV96].

**Theorem 6** (Treil–Volberg, 1995). *Let  $\mu$  be a Borel measure on  $\text{clos}(\mathbb{C}_+)$ , let  $\Theta$  be an inner function and let  $1 \leq p < +\infty$ . Assume that there is  $C > 0$  such that*

$$(9) \quad \mu(S(x, h)) \leq Ch,$$

*for every square  $S(x, h)$  satisfying  $S(x, h) \cap \Omega(\Theta, \varepsilon) \neq \emptyset$ . Then  $K_\Theta^p$  embeds continuously in  $L^p(\mu)$ .*

Moreover they showed that for the case where  $\Theta$  satisfies the connected level set condition, then the sufficient condition (9) is also necessary. In [Al99], Aleksandrov proved that the condition of Treil–Volberg is necessary if and only if  $\Theta \in CLS$ . Moreover, if  $\Theta$  does not satisfy the connected level set condition, then the class of measures  $\mu$  such that the inequality (7) is valid depend essentially on the exponent  $p$  (in contrast to the classical theorem of Carleson).

Of special interest is the case when  $\mu = \sum_{n \in \mathbb{N}} a_n \delta_{\{\lambda_n\}}$  is a discrete measure; then embedding is equivalent to the Bessel property for the system of reproducing kernels  $\{k_{\lambda_n}^\Theta\}$ .

Also the particular case when  $\mu$  is a measure on the real line is of great importance. In contrast to the embeddings of the whole Hardy space  $H^p$  (note that Carleson measures on  $\mathbb{R}$  are measures with bounded density with respect to Lebesgue measure  $m$ ), the class of Borel measures  $\mu$  such that  $K_\Theta^p \subset L^p(\mu)$  always contains nontrivial examples of singular measures on  $\mathbb{R}$ ; in particular, for  $p = 2$ , the Clark measures [Cl72] for which the embeddings  $K_\Theta^2 \subset L^2(\mu)$  are isometric. The situation changes for  $p \neq 2$  as shown by Aleksandrov [Al89]. See also the nice survey by Poltoratski and Sarason on what they call Aleksandrov-Clark measures [PS06]. On the other hand, if  $\mu = wm$ ,  $w \in L^2(\mathbb{R})$ , then the embedding problem is related to the properties of the Toeplitz operator  $T_w$  (see [Co86b]).

In [Ba03, Ba05a], Baranov developed a new approach based on the (weighted norm) Bernstein inequalities and he got some extensions of Cohn and Volberg–Treil results. Compactness of the embedding operator  $K_\Theta^p \subset L^2(\mu)$  is also of interest and is considered in [Vo81, Co86b, CM03, Ba05a, Ba08].

Another important result in connection with  $K_I^p$ -spaces is that of Douglas, Shapiro and Shields ([DSS70], see also [CR00, Theorem 1.0.5]) and concerns pseudocontinuation. Recall that a function holomorphic in  $\mathbb{D}_e := \hat{\mathbb{C}} \setminus \text{clos } \mathbb{D} - \text{clos } E$  means the closure of a set  $E$  — is a pseudocontinuation of a function  $f$  meromorphic in  $\mathbb{D}$  if  $\psi$  vanishes at  $\infty$  and the outer nontangential limits of  $\psi$  on  $\mathbb{T}$  coincide with the inner nontangential limits of  $f$  on  $\mathbb{T}$  in almost every point of  $\mathbb{T}$ . Note that  $f \in K_I^2 = H^2 \cap I\overline{H_0^2}$  implies that  $f = I\overline{\psi}$  with  $\psi \in H_0^2$ . Then the meromorphic function  $f/I$  equals  $\overline{\psi}$  a.e.  $\mathbb{T}$ , and writing  $\psi(z) = \sum_{n \geq 1} b_n z^n$ , it is clear that  $\tilde{\psi}(z) := \sum_{n \geq 1} \overline{b_n}/z^n$  is a holomorphic function in  $\mathbb{D}_e$ , vanishing at  $\infty$ , and being equal to  $f/I$  almost everywhere on  $\mathbb{T}$  (in fact,  $\tilde{\psi} \in H^2(\mathbb{D}_e)$ ). The converse is also true: if  $f/I$  has a pseudocontinuation in  $\mathbb{D}_e$ , where  $f$  is a  $H^p$ -function and  $I$  some inner function  $I$ , then  $f$  is in  $K_I^p$ . This can be resumed this in the following result.

**Theorem 7** (Douglas-Shapiro-Shields, 1972). *Let  $I$  be an inner function. Then a function  $f \in H^p$  is in  $K_I^p$  if and only if  $f/I$  has a pseudocontinuation to a function in  $H^p(\mathbb{D}_e)$  which vanishes at infinity.*

Note that there are functions analytic on  $\mathbb{C}$  that do not admit a pseudocontinuation. An example of such a function is  $f(z) = e^z$  which has an essential singularity at infinity.

As already mentioned, we will be concerned with two generalizations of the backward shift invariant subspaces. One direction is to consider weighted versions of such spaces. The other direction is to replace the inner function by more general functions. The appropriate definition of  $K_I^2$  in this setting is that of de Branges-Rovnyak spaces (requiring that  $p = 2$ ).

Our aim is to discuss some of the above results in the context of these spaces. For analytic continuation it turns out that the conditions in both cases are quite similar to the original  $K_I^2$ -situation. However in the weighted situation some additional condition is needed. For boundary behaviour in points in the spectrum the situation changes. In the de Branges-Rovnyak spaces the Ahern-Clark condition generalizes naturally, whereas in weighted backward shift invariant



subspaces the situation is not clear and awaits further investigation. This will be illustrated in Example 4.1.

### 3. DE BRANGES-ROVNYAK SPACES

Let us begin with defining de Branges-Rovnyak spaces. We will be essentially concerned with the special case of Toeplitz operators. Recall that for  $\varphi \in L^\infty(\mathbb{T})$ , the Toeplitz operator  $T_\varphi$  is defined on  $H^2(\mathbb{D})$  by

$$T_\varphi(f) := P_+(\varphi f) \quad (f \in H^2(\mathbb{D})),$$

where  $P_+$  denotes the orthogonal projection of  $L^2(\mathbb{T})$  onto  $H^2(\mathbb{D})$ . Then, for  $\varphi \in L^\infty(\mathbb{T})$ ,  $\|\varphi\|_\infty \leq 1$ , the de Branges-Rovnyak space  $\mathcal{H}(\varphi)$ , associated with  $\varphi$ , consists of those  $H^2(\mathbb{D})$  functions which are in the range of the operator  $(Id - T_\varphi T_{\bar{\varphi}})^{1/2}$ . It is a Hilbert space when equipped with the inner product

$$\langle (Id - T_\varphi T_{\bar{\varphi}})^{1/2} f, (Id - T_\varphi T_{\bar{\varphi}})^{1/2} g \rangle_\varphi = \langle f, g \rangle_2,$$

where  $f, g \in H^2(\mathbb{D}) \ominus \ker (Id - T_\varphi T_{\bar{\varphi}})^{1/2}$ .

These spaces (and more precisely their general vector-valued version) appeared first in L. de Branges and J. Rovnyak [dBR66a, dBR66b] as universal model spaces for Hilbert space contractions. As a special case, when  $b = I$  is an inner function (that is  $|b| = |I| = 1$  a.e. on  $\mathbb{T}$ ), the operator  $(Id - T_I T_{\bar{I}})$  is an orthogonal projection and  $\mathcal{H}(I)$  becomes a closed (ordinary) subspace of  $H^2(\mathbb{D})$  which coincides with the model spaces  $K_I = H^2(\mathbb{D}) \ominus IH^2(\mathbb{D})$ . Thanks to the pioneering work of Sarason, e.g. [Sa89, Sa89, Sa94, Sa95], we know that de Branges-Rovnyak spaces play an important role in numerous questions of complex analysis and operator theory. We mention a recent paper by the second named author and Sarason and Seip [HSS04] who gave a characterization of surjectivity of Toeplitz operator the proof of which involves de Branges-Rovnyak spaces. We also refer to work of J. Shapiro [Sh01, Sh03] concerning the notion of angular derivative for holomorphic self-maps of the unit disk. See also a paper of J. Anderson and J. Rovnyak [AR06], where generalized Schwarz-Pick estimates are given and a paper of M. Jury [Ju07], where composition operators are studied by methods based on  $\mathcal{H}(b)$  spaces.

In what follows we will assume that  $b$  is in the unit ball of  $H^\infty$ . We recall here that since  $\mathcal{H}(b)$  is contained contractively in  $H^2$ , it is a reproducing kernel Hilbert space. More precisely, for all function  $f$  in  $\mathcal{H}(b)$  and every point  $\lambda$  in  $\mathbb{D}$ , we have

$$(10) \quad f(\lambda) = \langle f, k_\lambda^b \rangle_b,$$

where  $k_\lambda^b = (Id - T_b T_{\bar{b}})k_\lambda$ . Thus

$$k_\lambda^b(z) = \frac{1 - \overline{b(\lambda)}b(z)}{1 - \bar{\lambda}z}, \quad z \in \mathbb{D}.$$

We also recall that  $\mathcal{H}(b)$  is invariant under the backward shift operator and in the following, we denote by  $X$  the contraction  $X := S_{|\mathcal{H}(b)}^*$ . Its adjoint satisfies the important formula

$$X^*h = Sh - \langle h, S^*b \rangle_b b, \quad h \in \mathcal{H}(b).$$

In the case where  $b$  is inner, then  $X$  coincides with the so-called model operator of Sz.-Nagy–Foias which serves as a model to Hilbert space contractions (in fact, those contractions  $T$  which are  $C_0$  and with  $\partial_T = \partial_{T^*} = 1$ ; for the general case, the model operator is quite complicated).

Finally, let us recall that a point  $\lambda \in \overline{\mathbb{D}}$  is said to be regular (for  $b$ ) if either  $\lambda \in \mathbb{D}$  and  $b(\lambda) \neq 0$ , or  $\lambda \in \mathbb{T}$  and  $b$  admits an analytic continuation across a neighbourhood  $V_\lambda = \{z : |z - \lambda| < \varepsilon\}$  of  $\lambda$  with  $|b| = 1$  on  $V_\lambda \cap \mathbb{T}$ . The spectrum of  $b$ , denoted by  $\sigma(b)$ , is then defined as the complement in  $\overline{\mathbb{D}}$  of all regular points of  $b$ .

In this section we will summarize the results corresponding to Theorems 1 and 2 above in the setting of de Branges–Rovnyak spaces. It turns out that Moeller’s result remains valid in the setting of de Branges–Rovnyak spaces. Concerning the result by Ahern–Clark, it turns out that if we replace the inner function  $I$  by a general function  $b$  in the ball of  $H^\infty$ , meaning that  $b = Ib_0$  where  $b_0$  is now outer, then we have to add to condition (ii) in Theorem 2 the term corresponding to the absolutely continuous part of the measure:  $|\log |b_0||$ .

In [FM08a], E. Fricain and J. Mashreghi studied the continuity and analyticity of functions in the de Branges–Rovnyak spaces  $\mathcal{H}(b)$  on an open arc of  $\mathbb{T}$ . As we will see the theory bifurcates into two opposite cases depending on whether  $b$  is an extreme point of the unit ball of  $H^\infty(\mathbb{D})$  or not. Let us recall that if  $X$  is a linear space and  $S$  is a convex subset of  $X$ , then an element  $x \in S$  is called an extreme point of  $S$  if it is not a proper convex combination of any two distinct points in  $S$ . Then, it is well known (see [Du70, page 125]) that a function  $f$  is an extreme point of the unit ball of  $H^\infty(\mathbb{D})$  if and only if

$$\int_{\mathbb{T}} \log(1 - |f(\zeta)|) d\zeta = -\infty.$$

The following result is a generalization of Theorem 1 of Moeller.

**Theorem 8** (Fricain–Mashreghi, 2008). *Let  $b$  be in the unit ball of  $H^\infty(\mathbb{D})$  and let  $I$  be an open arc of  $\mathbb{T}$ . Then the following are equivalent:*

- (i)  $b$  has an analytic continuation across  $I$  and  $|b| = 1$  on  $I$ ;
- (ii)  $I$  is contained in the resolvent set of  $X^*$ ;
- (iii) any function  $f$  in  $\mathcal{H}(b)$  has an analytic continuation across  $I$ ;
- (iv) any function  $f$  in  $\mathcal{H}(b)$  has a continuous extension to  $\mathbb{D} \cup I$ ;
- (v)  $b$  has a continuous extension to  $\mathbb{D} \cup I$  and  $|b| = 1$  on  $I$ .

The equivalence of (i), (ii) and (iii) were proved in [Sa95, page 42] under the assumption that  $b$  is an extreme point. The contribution of Fricain–Mashreghi concerns the the last two points. The mere assumption of continuity implies analyticity and this observation has interesting application as we will see below. Note that this implication is true also in the weighted situation (see Theorem 18).

The proof of Theorem 8 is based on reproducing kernel of  $\mathcal{H}(b)$  spaces. More precisely, we use the fact that given  $\omega \in \mathbb{D}$ , then  $k_\omega^b = (Id - \bar{\omega}X^*)^{-1}k_0^b$  and thus

$$f(\omega) = \langle f, k_\omega^b \rangle_b = \langle f, (Id - \bar{\omega}X^*)^{-1}k_0^b \rangle_b,$$

for every  $f \in \mathcal{H}(b)$ . Another key point in the proof of Theorem 8 is the theory of Hilbert spaces contractions developped by Sz.-Nagy–Foias. Indeed, if  $b$  is an extreme point of the unit ball of  $H^\infty$ , then the characteristic function of the contraction  $X^*$  is  $b$  (see [Sa86]) and then we know that  $\sigma(X^*) = \sigma(b)$ . Therefore the proof of Theorem 8 differs from the proof of the original proof of Moeller for the inner case.

It is easy to see that condition (i) in the previous result implies that  $b$  is an extreme point of the unit ball of  $H^\infty$ . Thus, the continuity (or equivalently, the analytic continuation) of  $b$  or of the elements of  $\mathcal{H}(b)$  on the boundary completely depends on whether  $b$  is an extreme point or not. If  $b$  is not an extreme point of the unit ball of  $H^\infty(\mathbb{D})$  and if  $I$  is an open arc of  $\mathbb{T}$ , then there exists necessarily a function  $f \in \mathcal{H}(b)$  such that  $f$  has not a continuous extension to  $\mathbb{D} \cup I$ . On the opposite case, if  $b$  is an extreme point such that  $b$  has continuous extension to  $\mathbb{D} \cup I$  with  $|b| = 1$  on  $I$ , then all the functions  $f \in \mathcal{H}(b)$  are continuous on  $I$  (and even can be continued analytically across  $I$ ).

As in the inner case (see Ahern–Clark’s result, Theorem 2), it is natural to ask what happens in points which are in the spectrum and what kind of regularity can be expected there. In [FM08a], E. Fricain and J. Mashregi gave an answer to this question and this result generalizes the Ahern–Clark result.

**Theorem 9** (Fricain–Mashregi, 2008). *Let  $b$  be a point in the unit ball of  $H^\infty(\mathbb{D})$  and let*

$$(11) \quad b(z) = \gamma \prod_k \left( \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \bar{a}_k z} \right) \exp \left( - \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \right) \exp \left( \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log |b(\zeta)| dm(\zeta) \right)$$

*be its canonical factorization. Let  $\zeta_0 \in \mathbb{T}$  and let  $n$  be a non-negative integer. Then the following are equivalent.*

- (i) *for every function  $f \in \mathcal{H}(b)$ ,  $f(z), f'(z), \dots, f^{(n)}(z)$  have finite limits as  $z$  tends radially to  $\zeta_0$ ;*
- (ii)  *$\|\partial^n k_z^b / \partial \bar{z}^n\|_b$  is bounded as  $z$  tends radially to  $\zeta_0$ ;*
- (iii)  *$X^{*n} k_0^b$  belongs to the range of  $(Id - \bar{\zeta}_0 X^*)^{n+1}$ ;*
- (iv) *we have*

$$(12) \quad \sum_k \frac{1 - |a_k|^2}{|\zeta_0 - a_k|^{2n+2}} + \int_0^{2\pi} \frac{d\mu(e^{it})}{|\zeta_0 - e^{it}|^{2n+2}} + \int_0^{2\pi} \frac{|\log |b(e^{it})||}{|\zeta_0 - e^{it}|^{2n+2}} dm(e^{it}) < +\infty.$$

The proof of Theorem 9 is based on a generalization of technics of Ahern–Clark. However, we should mention that the general case is a little bit more complicated than the inner case. Indeed for the equivalence of (iii) and (iv) (which is the hard part of the proof), Ahern–Clark noticed that the condition (iii) is equivalent to the following interpolation problem: there exists  $k, g \in H^2$  such that

$$(1 - \bar{\zeta}_0 z)^{n+1} k(z) - N! z^n = b(z) g(z).$$

This reformulation, based on the orthogonal decomposition  $H^2 = \mathcal{H}(b) \ominus bH^2$ , is crucial in the proof of Ahern–Clark. In the general case, this is no longer true because  $\mathcal{H}(b)$  is not a closed subspace of  $H^2$  and we cannot have such an orthogonal decomposition. This induces a real difficulty that we can overcome using other arguments: In particular, we use (in the proof) the

fact that if  $\zeta_0$  satisfies the condition (12), then, for  $0 \leq j \leq 2n + 1$ , the limits

$$\lim_{r \rightarrow 1^-} b^{(j)}(r) \quad \text{and} \quad \lim_{R \rightarrow 1^+} b^{(j)}(R)$$

exist and are equal. Here we extend the function  $b$  outside the unit disk by the formula (11), which represents an analytic function for  $|z| > 1$ ,  $z \neq 1/\bar{a}_n$ . We denote this function also by  $b$  and it is easily verified that it satisfies

$$(13) \quad b(z) = \frac{1}{\overline{b(1/\bar{z})}}, \quad \forall z \in \mathbb{C}.$$

Maybe we should compare condition (iv) of Theorem 9 and condition (ii) of Theorem 8. For the question of analytic continuation through a point  $\zeta_0 \in \mathbb{T}$ , we impose that the operator  $Id - \bar{\zeta}_0 X^*$  is bijective (or onto which is equivalent because it is always one-to-one as noted in [Fr05, Lemma 2.2]) whereas for the question of the existence of radial limits for the derivative up to a given order  $N$ , we impose only that the range of the operator  $(Id - \bar{\zeta}_0 X^*)^{n+1}$  contains the function  $X^{*n} k_0^b$ . We also mention that Sarason has obtained another criterion in terms of the measure  $\mu_\lambda$  whose Poisson integral is the real part of  $\frac{\lambda + b}{\lambda - b}$  with  $\lambda \in \mathbb{T}$ .

**Theorem 10** (Sarason, 1995). *Let  $\zeta_0$  be a point of  $\mathbb{T}$  and let  $n$  be a nonnegative integer. The following conditions are equivalent.*

- (i) *Each function in  $\mathcal{H}(b)$  and all its derivatives up to order  $n$  have nontangential limits at  $z_0$ .*
- (ii) *There is a point  $\lambda \in \mathbb{T}$  such that*

$$\int_{\mathbb{T}} |e^{i\theta} - z_0|^{-2n-2} d\mu_\lambda(e^{i\theta}) < +\infty.$$

- (iii) *The last inequality holds for all  $\lambda \in \mathbb{T} \setminus \{b(z_0)\}$ .*
- (iv) *There is a point  $\lambda \in \mathbb{T}$  such that  $\mu_\lambda$  has a point mass at  $z_0$  and*

$$\int_{\mathbb{T} \setminus \{z_0\}} |e^{i\theta} - z_0|^{-2n} d\mu_\lambda(e^{i\theta}) < \infty.$$

Recently, Bolotnikov and Kheifets [BK06] gave a third criterion (in some sense more algebraic) in terms of the Schwarz-Pick matrix. Recall that if  $b$  is a function in the unit ball of  $H^\infty$ , then the matrix  $\mathbf{P}_n^\omega(z)$ , which will be referred to as to a Schwarz-Pick matrix and defined by

$$\mathbf{P}_n^b(z) := \left[ \frac{1}{i!j!} \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} \frac{1 - |b(z)|^2}{1 - |z|^2} \right]_{i,j=0}^n,$$

is positive semidefinite for every  $n \geq 0$  and  $z \in \mathbb{D}$ . We extend this notion to boundary points as follows: given a point  $z_0 \in \mathbb{T}$ , the boundary Schwarz-Pick matrix is

$$\mathbf{P}_n^b(\zeta_0) = \lim_{\substack{z \rightarrow z_0 \\ \triangleleft}} \mathbf{P}_n^b(z) \quad (n \geq 0),$$

provided this non tangential limit exists.

**Theorem 11** (Bolotnikov–Kheifets, 2006). *Let  $b$  be a point in the unit ball of  $H^\infty$ , let  $z_0 \in \mathbb{T}$  and let  $n$  be a nonnegative integer. Assume that the boundary Schwarz-Pick matrix  $\mathbf{P}_n^b(z_0)$  exists. Then each function in  $\mathcal{H}(b)$  and all its derivatives up to order  $n$  have nontangential limits at  $z_0$ .*

Further it is shown in [BK06] that the boundary Schwarz-Pick matrix  $\mathbf{P}_n^b(z_0)$  exists if and only if

$$(14) \quad \lim_{\substack{z \rightarrow z_0 \\ <}} d_{b,n}(z) < +\infty,$$

where

$$d_{b,n}(z) := \frac{1}{(n!)^2} \frac{\partial^{2n}}{\partial z^n \partial \bar{z}^n} \frac{1 - |b(z)|^2}{1 - |z|^2}.$$

We should mention that it is not clear to show direct connections between conditions (12), (14) and (ii)–(iv) of Theorem 10.

Theorems 9, 10 and 11 have of course an analogue in the setting of the upper half plane  $\mathbb{C}_+$ . In the following,  $\Theta$  will denote a function in the unit ball of  $H^\infty(\mathbb{C}_+)$  and  $\mathcal{H}(\Theta)$  will be the associated de Branges–Rovnyak spaces (the definition of de Branges–Rovnyak spaces of the upper half-plane is similar to its counterpart for the unit disc). Once we know the points  $x_0$  of the real line where  $f^{(n)}(x_0)$  exists (in a non-tangential sense) for every function  $f \in \mathcal{H}(\Theta)$ , it is natural to ask if we can obtain an integral formulae for this derivative similar to (2) for the inner case. However, if one tries to generalize techniques used in the model spaces  $K_\Theta^2$  in order to obtain such a representation for the derivatives of functions in  $\mathcal{H}(\Theta)$ , some difficulties appear mainly due to the fact that the evaluation functional in  $\mathcal{H}(\Theta)$  (contrary to the model space  $K_\Theta^2$ ) is not a usual integral operator. To overcome this difficulty and nevertheless provide an integral formula similar to (2) for functions in  $\mathcal{H}(\Theta)$ , Fricain–Mashreghi used in [FM08b] two general facts about the de Branges–Rovnyak spaces that we recall now. The first one concerns the relation between  $\mathcal{H}(\Theta)$  and  $\mathcal{H}(\bar{\Theta})$ . For  $f \in H^2(\mathbb{C}_+)$ , we have [Sa95, page 10]

$$f \in \mathcal{H}(\Theta) \iff T_{\bar{\Theta}} f \in \mathcal{H}(\bar{\Theta}).$$

Moreover, if  $f_1, f_2 \in \mathcal{H}(\Theta)$ , then

$$(15) \quad \langle f_1, f_2 \rangle_\Theta = \langle f_1, f_2 \rangle_2 + \langle T_{\bar{\Theta}} f_1, T_{\bar{\Theta}} f_2 \rangle_{\bar{\Theta}}.$$

We also mention an integral representation for functions in  $\mathcal{H}(\bar{\Theta})$  [Sa95, page 16]. Let  $\rho(t) := 1 - |\Theta(t)|^2$ ,  $t \in \mathbb{R}$ , and let  $L^2(\rho)$  stand for the usual Hilbert space of measurable functions  $f : \mathbb{R} \rightarrow \mathbb{C}$  with  $\|f\|_\rho < \infty$ , where

$$\|f\|_\rho^2 := \int_{\mathbb{R}} |f(t)|^2 \rho(t) dt.$$

For each  $w \in \mathbb{C}_+$ , the Cauchy kernel  $k_w$  belongs to  $L^2(\rho)$ . Hence, we define  $H^2(\rho)$  to be the span in  $L^2(\rho)$  of the functions  $k_w$  ( $w \in \mathbb{C}_+$ ). If  $q$  is a function in  $L^2(\rho)$ , then  $q\rho$  is in  $L^2(\mathbb{R})$ , being the product of  $q\rho^{1/2} \in L^2(\mathbb{R})$  and the bounded function  $\rho^{1/2}$ . Finally, we define the operator  $C_\rho : L^2(\rho) \rightarrow H^2(\mathbb{C}_+)$  by

$$C_\rho(q) := P_+(q\rho).$$

Then  $C_\rho$  is a partial isometry from  $L^2(\rho)$  onto  $\mathcal{H}(\bar{\Theta})$  whose initial space equals to  $H^2(\rho)$  and it is an isometry if and only if  $\Theta$  is an extreme point of the unit ball of  $H^\infty(\mathbb{C}_+)$ .

Now let  $\omega \in \overline{\mathbb{C}_+}$  and let  $n$  be a non-negative integer. In order to get an integral representation for the  $n$ th derivative of  $f$  at point  $\omega$  for functions in the de-Branges-Rovnyak spaces, we need to introduce the following kernels

$$(16) \quad k_{\omega,n}^\Theta(z) := \frac{1 - \Theta(z) \sum_{p=0}^n \frac{\overline{\Theta^{(p)}(\omega)}}{p!} (z - \bar{\omega})^p}{(z - \bar{\omega})^{n+1}}, \quad (z \in \mathbb{C}_+),$$

and

$$(17) \quad k_{\omega,n}^\rho(t) := \frac{\sum_{p=0}^n \frac{\overline{\Theta^{(p)}(\omega)}}{p!} (t - \bar{\omega})^p}{(t - \bar{\omega})^{n+1}}, \quad (t \in \mathbb{R}).$$

Of course, for  $\omega = x_0 \in \mathbb{R}$ , these formulae have a sense only if  $\Theta$  has derivatives (in a radial or nontangential sense) up to order  $n$ ; as we have seen this is the case if  $x_0$  satisfies the condition

$$(18) \quad \sum_k \frac{\operatorname{Im}(z_k)}{|x_0 - z_k|^p} + \int_{\mathbb{R}} \frac{d\mu(t)}{|x_0 - t|^p} + \int_{\mathbb{R}} \frac{|\log |\Theta(t)||}{|x_0 - t|^p} dt < +\infty,$$

for  $p = n + 1$ , where  $(z_k)_k$  is the sequence of the zeros of  $\Theta$ , and  $\mu$  is the singular measure associated to  $\Theta$  in its canonical factorization. The condition (18) is the analogue of (12) for the upper half-plane. In the following, we denote by  $E_p(\Theta)$  the set of real points  $x_0$  which satisfy the condition (18).

For  $n = 0$ , we see that  $k_{\omega,0}^\Theta = k_\omega^\Theta$  is the reproducing kernel of  $\mathcal{H}(\Theta)$  and  $k_{\omega,0}^\rho = \overline{\Theta(\omega)} k_\omega$  is (up to a constant) the Cauchy kernel. Moreover (at least formally) the function  $k_{\omega,n}^\Theta$  (respectively  $k_{\omega,n}^\rho$ ) is the  $n$ -th derivative of  $k_{\omega,0}^\Theta$  (respectively of  $k_{\omega,0}^\rho$ ) with respect to  $\bar{\omega}$ .

**Theorem 12** (Fricain–Mashreghi, 2008). *Let  $\Theta$  be a function in the unit ball of  $H^\infty(\mathbb{C}_+)$  and let  $n$  be a non-negative integer. Then for every point  $z_0 \in \mathbb{C}_+ \cup E_{2n+2}(\Theta)$  and for every function  $f \in \mathcal{H}(\Theta)$ , we have  $k_{z_0,n}^\Theta \in \mathcal{H}(\Theta)$ ,  $k_{z_0,n}^\rho \in L^2(\rho)$  and*

$$(19) \quad f^{(n)}(z_0) = \frac{n!}{2i\pi} \left( \int_{\mathbb{R}} f(t) \overline{k_{z_0,n}^\Theta(t)} dt + \int_{\mathbb{R}} g(t) \rho(t) \overline{k_{z_0,n}^\rho(t)} dt \right),$$

where  $g \in H^2(\rho)$  satisfies  $T_\Theta f = C_\rho g$ .

We see that if  $\Theta$  is inner, then it is clear that the second integral in (19) is zero (because  $\rho \equiv 0$ ) and we obtain the formula (2) of Ahern–Clark (more precisely the analogue in the upper half-plane). We should mention that in the case where  $z_0 \in \mathbb{C}_+$ , the formula (19) follows easily from the formulae (15) and (10). For  $z_0 \in E_n(\Theta)$ , the result is more delicate and the key point of the proof is to show that

$$(20) \quad f^{(n)}(z_0) = \langle f, k_{z_0,n}^\Theta \rangle_\Theta,$$

for every function  $f \in \mathcal{H}(\Theta)$  and then show that  $T_\Theta k_{z_0,n}^\Theta = C_\rho k_{z_0,n}^\rho$  to use once again (15).

A consequence of (20) and Theorem 9 is that if  $z_0 \in E_{2n+2}(\Theta)$ , then  $k_{\omega,n}^\Theta$  tends weakly to  $k_{z_0,n}^\Theta$  as  $\omega$  approaches radially to  $x_0$ . It is natural to ask if this weak convergence can be replaced by norm convergence. In other words, is it true that  $\|k_{\omega,n}^\Theta - k_{x_0,n}^\Theta\|_\Theta \rightarrow 0$  as  $\omega$  tends radially to  $x_0$ ?

In [AC70], Ahern and Clark claimed that they can prove this result for the case where  $\Theta$  is inner and  $n = 0$ . For general functions  $\Theta$  in the unit ball of  $H^\infty(\mathbb{C}_+)$ , Sarason [Sa95, Chap. V] got this norm convergence for the case  $n = 0$ . In [FM08b], we answer this question in the general case and get the following result.

**Theorem 13** (Fricain–Mashreghi, 2008). *Let  $\Theta$  be a point in the unit ball of  $H^\infty(\mathbb{C}_+)$ , let  $n$  be a non-negative integer and let  $x_0 \in \mathbb{R}$  satisfying the condition (18). Then*

$$\|k_{\omega,n}^\Theta - k_{x_0,n}^\Theta\|_\Theta \longrightarrow 0, \quad \text{as } \omega \text{ tends radially to } x_0.$$

The proof is based on explicit computations of  $\|k_{\omega,n}^\Theta\|_\Theta$  and  $\|k_{x_0,n}^\Theta\|_\Theta$  and we use a non trivial formula of combinatorics for sums of binomial coefficient. We should mention that we have obtained this formula by hypergeometric series. Let us also mention that Bolotnikov–Kheifets got a similar result in [BK06] using different techniques and under their condition (14).

We will now discuss the weighted norm inequalities obtained in [BFM09]. The main goal was to get an analogue of Theorem 4 in the setting of the de Branges–Rovnyak spaces. To get these weighted Bernstein type inequalities, we first used a slight modified formula of (19).

**Proposition 1** (Baranov–Fricain–Mashreghi, 2009). *Let  $\Theta$  be in the unit ball of  $H^\infty(\mathbb{C}_+)$ . Let  $z_0 \in \mathbb{C}_+ \cup E_{2n+2}(\Theta)$ ,  $n \in \mathbb{N}$ , and let*

$$(21) \quad \mathfrak{K}_{z_0,n}^\rho(t) := \overline{\Theta(z_0)} \frac{\sum_{j=0}^n \binom{n+1}{j+1} (-1)^j \overline{\Theta^j(z_0)} \Theta^j(t)}{(t - \overline{z_0})^{n+1}}, \quad t \in \mathbb{R}.$$

*Then  $(k_{z_0}^\Theta)^{n+1} \in H^2(\mathbb{C}_+)$  and  $\mathfrak{K}_{z_0,n}^\rho \in L^2(\rho)$ . Moreover, for every function  $f \in \mathcal{H}(\Theta)$ , we have*

$$(22) \quad f^{(n)}(z_0) = \frac{n!}{2\pi i} \left( \int_{\mathbb{R}} f(t) \overline{(k_{z_0}^\Theta)^{n+1}(t)} dt + \int_{\mathbb{R}} g(t) \rho(t) \overline{\mathfrak{K}_{z_0,n}^\rho(t)} dt \right),$$

*where  $g \in H^2(\rho)$  is such that  $T_\Theta f = C_\rho g$ .*

We now introduce the weight involved in our Bernstein-type inequalities. Let  $1 < p \leq 2$  and let  $q$  be its conjugate exponent. Let  $n \in \mathbb{N}$ . Then, for  $z \in \overline{\mathbb{C}_+}$ , we define

$$w_{p,n}(z) := \min \left\{ \|(k_z^\Theta)^{n+1}\|_q^{-pn/(pn+1)}, \|\rho^{1/q} \mathfrak{K}_{z,n}^\rho\|_q^{-pn/(pn+1)} \right\};$$

we assume  $w_{p,n}(x) = 0$ , whenever  $x \in \mathbb{R}$  and at least one of the functions  $(k_x^\Theta)^{n+1}$  or  $\rho^{1/q} \mathfrak{K}_{x,n}^\rho$  is not in  $L^q(\mathbb{R})$ .

The choice of the weight is motivated by the representation (22) which shows that the quantity  $\max \{ \|(k_z^\Theta)^{n+1}\|_2, \|\rho^{1/2} \mathfrak{K}_{z,n}^\rho\|_2 \}$  is related to the norm of the functional  $f \mapsto f^{(n)}(z)$  on  $\mathcal{H}(\Theta)$ . Moreover, we strongly believe that the norms of reproducing kernels are an important characteristic of the space  $\mathcal{H}(\Theta)$  which captures many geometric properties of  $\Theta$ . Using similar arguments as in the proof of proposition 1, it is easy to see that  $\rho^{1/q} \mathfrak{K}_{x,n}^\rho \in L^q(\mathbb{R})$  if  $x \in E_{q(n+1)}(\Theta)$ . It

is also natural to expect that  $(k_x^\Theta)^{n+1} \in L^q(\mathbb{R})$  for  $x \in E_{q(n+1)}(\Theta)$ . This is true when  $b$  is an inner function, by a result of Cohn [Co86a]; for a general function  $\Theta$  with  $q = 2$  it was noticed in [BFM09]. However, it seems that the methods of [Co86a] and [BFM09] do not apply in the general case.

If  $f \in \mathcal{H}(\Theta)$  and  $1 < p \leq 2$ , then  $(f^{(n)}w_{p,n})(x)$  is well-defined on  $\mathbb{R}$ . Indeed it follows from [FM08a] that  $f^{(n)}(x)$  and  $w_{p,n}(x)$  are finite if  $x \in E_{2n+2}(\Theta)$ . On the contrary if  $x \notin E_{2n+2}(\Theta)$ , then  $\|(k_x^\Theta)^{n+1}\|_2 = +\infty$ . Hence,  $\|(k_x^\Theta)^{n+1}\|_q = +\infty$  which, by definition, implies  $w_{p,n}(x) = 0$ , and thus we may assume  $(f^{(n)}w_{p,n})(x) = 0$ .

In the inner case, we have  $\rho(t) \equiv 0$  and the second term in the definition of the weight  $w_{p,n}$  disappears. It should be emphasized that in the general case both terms are essential: in [BFM09] we give an example where the norm  $\|\rho^{1/q}\mathfrak{K}_{z,n}^\rho\|_q$  cannot be majorized uniformly by the norm  $\|(k_z^\Theta)^{n+1}\|_q$ .

**Theorem 14** (Baranov–Fricain–Mashreghi, 2009). *Let  $\mu$  be a Carleson measure on  $\text{clos}(\mathbb{C}_+)$ , let  $n \in \mathbb{N}$ , let  $1 < p \leq 2$ , and let*

$$(T_{p,n}f)(z) = f^{(n)}(z)w_{p,n}(z), \quad f \in \mathcal{H}(\Theta).$$

*If  $1 < p < 2$ , then  $T_{p,n}$  is a bounded operator from  $\mathcal{H}(\Theta)$  to  $L^2(\mu)$ , that is, there is a constant  $C = C(\mu, p, n) > 0$  such that*

$$(23) \quad \|f^{(n)}w_{p,n}\|_{L^2(\mu)} \leq C\|f\|_b, \quad f \in \mathcal{H}(\Theta).$$

*If  $p = 2$ , then  $T_{2,n}$  is of weak type  $(2, 2)$  as an operator from  $\mathcal{H}(\Theta)$  to  $L^2(\mu)$ .*

The proof of this result is based on the representation (22) which reduces the problem of Bernstein type inequalities to estimates on singular integrals. In particular, we use the following estimates on the weight: for  $1 < p \leq 2$  and  $n \in \mathbb{N}$ , there exists a constant  $A = A(n, p) > 0$  such that

$$w_{p,n}(z) \geq A \frac{(\text{Im } z)^n}{(1 - |\Theta(z)|)^{\frac{pn}{q(pn+1)}}}, \quad z \in \mathbb{C}_+.$$

To apply Theorem 14 one should have effective estimates for the weight  $w_{p,n}$ , that is, for the norms of the reproducing kernels. In the following, we relate the weight  $w_{p,n}$  to the distances to the level sets of  $|\Theta|$ . We start with some notations. Denote by  $\sigma_i(\Theta)$  the boundary spectrum of  $\Theta$ , i.e.

$$\sigma_i(\Theta) := \left\{ x \in \mathbb{R} : \liminf_{\substack{z \rightarrow x \\ z \in \mathbb{C}_+}} |\Theta(z)| < 1 \right\}.$$

Then  $\text{clos } \sigma_i(\Theta) = \sigma(\Theta) \cap \mathbb{R}$  where  $\sigma(\Theta)$  is the spectrum defined at the begining of this section. For  $\varepsilon \in (0, 1)$ , we put

$$\Omega(\Theta, \varepsilon) := \{z \in \mathbb{C}_+ : |\Theta(z)| < \varepsilon\},$$

and

$$\tilde{\Omega}(\Theta, \varepsilon) := \sigma_i(\Theta) \cup \Omega(\Theta, \varepsilon).$$



Finally, for  $x \in \mathbb{R}$ , we introduce the following three distances

$$\begin{aligned} d_0(x) &:= \text{dist}(x, \sigma_i(b)), \\ d_\varepsilon(x) &:= \text{dist}(x, \Omega(b, \varepsilon)), \\ \tilde{d}_\varepsilon(x) &:= \text{dist}(x, \tilde{\Omega}(b, \varepsilon)). \end{aligned}$$

Note that whenever  $b = \Theta$  is an inner function, for all  $x \in \sigma_i(\Theta)$ , we have

$$\liminf_{\substack{z \xrightarrow{x} \\ z \in \mathbb{C}_+}} |\Theta(z)| = 0,$$

and thus  $d_\varepsilon(t) = \tilde{d}_\varepsilon(t)$ ,  $t \in \mathbb{R}$ . However, for an arbitrary function  $\Theta$  in the unit ball of  $H^\infty(\mathbb{C}_+)$ , we have to distinguish between the distance functions  $d_\varepsilon$  and  $\tilde{d}_\varepsilon$ .

Using fine estimates on the derivatives  $|\Theta'(x)|$ , we got in [BFM09] the following result.

**Lemma 1.** *For each  $p > 1$ ,  $n \geq 1$  and  $\varepsilon \in (0, 1)$ , there exists  $C = C(\varepsilon, p, n) > 0$  such that*

$$(24) \quad (\tilde{d}_\varepsilon(x))^n \leq C w_{p,n}(x + iy),$$

for all  $x \in \mathbb{R}$  and  $y \geq 0$ .

This lemma combined with Theorem 14 imply immediately the following.

**Corollary 1** (Baranov–Fricain–Mashregi, 2009). *For each  $\varepsilon \in (0, 1)$  and  $n \in \mathbb{N}$ , there exists  $C = C(\varepsilon, n)$  such that*

$$\|f^{(n)} \tilde{d}_\varepsilon^n\|_2 \leq C \|f\|_\Theta, \quad f \in \mathcal{H}(\Theta).$$

As we have said in section 2, weighted Bernstein-type inequalities of the form (23) turned out to be an efficient tool for the study of the so-called Carleson-type embedding theorems for the shift-coinvariant subspaces  $K_\Theta^p$ . Notably, methods based on the Bernstein-type inequalities allow to give unified proofs and essentially generalize almost all known results concerning these problems (see [Ba05a, Ba08]). Here we obtain an embedding theorem for de Branges–Rovnyak spaces. In the case of an inner function the first statement coincides with a well-known theorem due to Volberg and Treil [TV96].

**Theorem 15.** *Let  $\mu$  be a Borel measure in  $\overline{\mathbb{C}_+}$ , and let  $\varepsilon \in (0, 1)$ .*

(a) *Assume that  $\mu(S(x, h)) \leq Kh$  for all Carleson squares  $S(x, h)$  satisfying*

$$S(x, h) \cap \tilde{\Omega}(\Theta, \varepsilon) \neq \emptyset.$$

*Then  $\mathcal{H}(\Theta) \subset L^2(\mu)$ , that is, there is a constant  $C > 0$  such that*

$$\|f\|_{L^2(\mu)} \leq C \|f\|_\Theta, \quad f \in \mathcal{H}(\Theta).$$

(b) *Assume that  $\mu$  is a vanishing Carleson measure for  $\mathcal{H}(\Theta)$ , that is,  $\mu(S(x, h))/h \rightarrow 0$  whenever  $S(x, h) \cap \tilde{\Omega}(b, \varepsilon) \neq \emptyset$  and  $h \rightarrow 0$  or  $\text{dist}(S(x, h), 0) \rightarrow +\infty$ . Then the embedding  $\mathcal{H}(\Theta) \subset L^2(\mu)$  is compact.*

In Theorem 15 we need to verify the Carleson condition only on a *special* subclass of squares. Geometrically this means that when we are far from the spectrum  $\sigma(\Theta)$ , the measure  $\mu$  in Theorem 15 can be essentially larger than standard Carleson measures. The reason is that functions in  $\mathcal{H}(\Theta)$  have much more regularity at the points  $x \in \mathbb{R} \setminus \text{clos } \sigma(\Theta)$  where  $|\Theta(x)| = 1$ . On the other hand, if  $|\Theta(x)| \leq \delta < 1$ , almost everywhere on some interval  $I \subset \mathbb{R}$ , then the functions in  $\mathcal{H}(\Theta)$  behave on  $I$  essentially the same as a general element of  $H^2(\mathbb{C}_+)$  on that interval, and for any Carleson measure for  $\mathcal{H}(\Theta)$  its restriction to the square  $S(I)$  is a standard Carleson measure.

For a class of functions  $\Theta$  the converse to Theorem 15 is also true. As in the inner case, we say that  $\Theta$  satisfies the *connected level set condition* if the set  $\Omega(\Theta, \varepsilon)$  is connected for some  $\varepsilon \in (0, 1)$ . Our next result is analogous to certain results from [Co82] and to [TV96, Theorem 3].

**Theorem 16.** *Let  $\Theta$  satisfy the connected level set condition for some  $\varepsilon \in (0, 1)$ . Assume that  $\Omega(\Theta, \varepsilon)$  is unbounded and  $\sigma(\Theta) \subset \text{clos } \Omega(\Theta, \varepsilon)$ . Let  $\mu$  be a Borel measure on  $\overline{\mathbb{C}_+}$ . Then the following statements are equivalent:*

- (a)  $\mathcal{H}(\Theta) \subset L^2(\mu)$ .
- (b) *There exists  $C > 0$  such that  $\mu(S(x, h)) \leq Ch$  for all Carleson squares  $S(x, h)$  such that  $S(x, h) \cap \widetilde{\Omega}(\Theta, \varepsilon) \neq \emptyset$ .*
- (c) *There exists  $C > 0$  such that*

$$(25) \quad \int_{\overline{\mathbb{C}_+}} \frac{\text{Im } z}{|\zeta - \bar{z}|^2} d\mu(\zeta) \leq \frac{C}{1 - |\Theta(z)|}, \quad z \in \mathbb{C}_+.$$

In [BFM09], we also discuss another application of our Bernstein type inequalities to the problem of stability of Riesz bases consisting of reproducing kernels in  $\mathcal{H}(\Theta)$ .

#### 4. WEIGHTED BACKWARD SHIFT INVARIANT SUBSPACES

Let us now turn to weighted backward shift invariant subspaces. In Subsection 4.1 we will present an example showing that the generalization of the Ahern-Clark result to the weighted situation is far from being immediate. For this reason we will focus on analytic continuation in this section.

For an outer function  $g$  in  $H^p$ , we define weighted Hardy spaces in the following way:

$$\begin{aligned} H^p(|g|^p) &:= \frac{1}{g} H^p = \{f \in \text{Hol}(\mathbb{D}) : \|f\|_{|g|^p}^p = \sup_{0 < r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^p |g(re^{it})|^p dt \\ &= \int_{-\pi}^{\pi} |f(e^{it})|^p |g(e^{it})|^p dt < \infty\}. \end{aligned}$$

Clearly  $f \mapsto fg$  induces an isometry from  $H^p(|g|^p)$  onto  $H^p$ . Let now  $I$  be any inner function.

We shall discuss the situation when  $p = 2$ . There are at least two ways of generalizing the backward shift invariant subspaces to the weighted situation. We first discuss the simple one. As in the unweighted situation we can consider the orthogonal complement of shift invariant subspaces  $IH^2(|g|^2)$ , the shift  $S : H^2(|g|^2) \rightarrow H^2(|g|^2)$  being given as usual by  $Sf(z) =$

$zf(z)$ . The weighted scalar product is defined by

$$\langle f, h \rangle_{|g|^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) \overline{h(e^{it})} |g(e^{it})|^2 dt = \langle fg, hg \rangle.$$

Then

$$\langle Sf, h \rangle_{|g|^2} = \langle zfg, hg \rangle = \langle fg, \bar{z}hg \rangle = \langle fg, P_+(\bar{z}hg) \rangle = \langle f, \frac{1}{g} P_+(g\bar{z}h) \rangle_{|g|^2}.$$

In other words, with respect to the scalar product  $\langle \cdot, \cdot \rangle_{|g|^2}$  the adjoint shift is given by  $S_g^* := \frac{1}{g} P_+ g \bar{z}$ , and

$$\begin{aligned} K_I^{2,g} &:= (IH^2(|g|^2))^\perp = \{f \in H^2(|g|^2) : \langle fg, Ihg \rangle = 0, h \in H^2(|g|^2)\} \\ &= \{f \in H^2(|g|^2) : \langle fg, Ih \rangle = 0, h \in H^2\} \\ &= \{f \in H^2(|g|^2) : \langle P_+(\bar{I}fg), h \rangle = 0, h \in H^2\} \\ &= \{f \in H^2(|g|^2) : \langle \frac{1}{g} P_+ \bar{I}fg, h \rangle_{|g|^2} = 0, h \in H^2(|g|^2)\} \end{aligned}$$

So,  $K_I^{2,g} = \ker(\frac{1}{g} P_+ \bar{I}g) = \ker(\frac{1}{g} P_+ \bar{I}g)$ . Setting  $P_I^g := \frac{1}{g} P_+ \bar{I}g$  we get a selfadjoint projection such that

$$K_I^{2,g} = P_I^g H^2(|g|^2) = \frac{1}{g} P_I (g H^2(|g|^2)) = \frac{1}{g} P_I H^2 = \frac{1}{g} K_I^2,$$

where  $P_I$  is the unweighted orthogonal projection onto  $K_I^2$ . Hence, in this situation continuation is completely determined by that in  $K_I^2$  and that of  $1/g$ .

Note also that the following simple example shows that in general  $K_I^{2,g}$  is different from  $K_I^2(|g|^2)$ . Let  $I(z) = z$  be the simplest Blaschke factor. Then  $H^2(|g|^2) \cap \overline{IH_0^2(|g|^2)} = H^2(|g|^2) \cap \overline{H^2(|g|)} = \mathbb{C}$  whenever  $g$  is rigid (more on rigidity follows later). On the other hand,  $\frac{1}{g} K_I^2$  is the one-dimensional space spanned by  $1/g$  which is different from  $\mathbb{C}$  when  $g$  is not a constant.

We will thus rather consider the second approach. The spaces to be discussed now appear in the context of kernels of Toeplitz operators. Set

$$K_I^p(|g|^p) = H^p(|g|^p) \cap \overline{IH_0^p(|g|^p)},$$

where now  $H^p(|g|^p) = zH^p(|g|^p)$ .

The connection with Toeplitz operators arises in the following way: if  $\varphi = \bar{I}g/g$  is a unimodular symbol, then  $\ker T_\varphi = gK_I^2(|g|^2)$  (see [HS03]). Conversely, whenever  $0 \neq f \in \ker T_\varphi$ , where  $\varphi$  is unimodular and  $f = Jg$  is the inner-outer factorization of  $f$ , then there exists an inner function  $I$  such that  $\varphi = \bar{I}g/g$  (see also [HS03]).

The representation  $\ker T_\varphi = gK_I^p(|g|^p)$  is particularly interesting when  $g$  is the extremal function of  $\ker T_\varphi$ . Then we know from a result by Hitt [Hi88] (see also [Sa94] for a de Branges-Rovnyak spaces approach to Hitt's result) that when  $p = 2$ ,  $\ker T_\varphi = gK_I^2$ , and that  $g$  is an isometric divisor on  $\ker T_\varphi = gK_I^2$  (or  $g$  is an isometric multiplier on  $K_I^2$ ). In this situation we thus have  $K_I^2(|g|^2) = K_I^2$ . Note, that for  $p \neq 2$ , if  $g$  is extremal for  $gK_I^p(|g|^p)$ , then  $K_I^p(|g|^p)$  can still be imbedded into  $K_I^2$  when  $p > 2$  and in  $K_I^p$  when  $p \in (1, 2)$  (see [HS03], where it is also

shown that these imbeddings can be strict). In these situations when considering questions concerning pseudocontinuation and analytic continuation, we can carry over to  $K_I^p(|g|^2)$  everything we know about  $K_I^2$  or  $K_I^p$ , i.e. Theorems 1 and 7. Concerning the Ahern-Clark and Cohn results however, when  $p \neq 2$ , we lose information since condition (ii) in Theorem 2 depends on  $p$ .

In general the extremal function is not easily detectable (explicit examples of extremal functions were given in [HS03]), in that we cannot determine it, or for a given  $g$  it is not a simple matter to check whether it is extremal or not. So a natural question is to know under which conditions on  $g$  and  $I$ , we can still say something about analytic continuation of functions in  $K_I^p(|g|^p)$ . It turns out that Moeller's result is valid under an additional local integrability condition of  $1/g$  on a closed arc not meeting the spectrum of  $I$ . Concerning the regularity questions in points contained in the spectrum, the situation is more intricate. As mentioned earlier, an example in this direction will be discussed at the end of this section.

Regularity of functions in kernels of Toeplitz operators have been considered by Dyakonov. He in particular establishes global regularity properties of functions in the kernel of a Toeplitz operator — such as being in certain Sobolev and Besov spaces [Dy96] or Lipschitz and Zygmund spaces [Dy08b] — depending on the smoothness of the corresponding Toeplitz operator.

The following simple example hints at some difference between this situation and the unweighted situation or the context of de Branges-Rovnyak spaces discussed before. Let  $I$  be arbitrary with  $-1 \notin \sigma(I)$ , and let  $g(z) = 1 + z$ , so that  $\sigma(I)$  is far from the only point where  $g$  vanishes. We know that  $\ker T_{\frac{1}{g}} = gK_I^p(|g|^p)$ . We first observe that  $\frac{1+z}{1+\bar{z}} = \bar{z}$ . Hence,

$$1 \in K_{zI}^p = \ker T_{zI} = \ker T_{\frac{1}{g}} = gK_I^p(|g|^p)$$

So,  $K_I^p(|g|^p)$  contains the function  $1/g$  which is badly behaved in  $-1$ , and thus cannot extend analytically through  $-1$ .

This observation can be made more generally as stated in the following result [H08].

**Proposition 2** (Hartmann 2008). *Let  $g$  be an outer function in  $H^p$ . If  $\ker T_{\bar{g}/g} \neq \{0\}$  contains an inner function, then  $1/g \in K_I^p(|g|^p)$  for every inner function  $I$ .*

Note that if the inner function  $J$  is in  $\ker T_{\bar{g}/g}$  then  $T_{\bar{g}/g}1 = 0$ , and hence  $1 \in \ker T_{\bar{g}/g} = gK_J^p(|g|^2)$  and  $1/g \in K_J^p(|g|^2)$ , which shows that with this simple argument the proposition holds with the more restrictive condition  $I = J$ .

Let us comment on the case  $p = 2$ :

The claim that the kernel of  $T_{\bar{g}/g}$  contains an inner function implies in particular that  $T_{\bar{g}/g}$  is not injective and so  $g^2$  is not rigid in  $H^1$  (see [Sa95, X-2]), which means that it is not uniquely determined — up to a real multiple — by its argument (or equivalently, its normalized version  $g^2/\|g^2\|_1$  is not exposed in the unit ball of  $H^1$ ).

It is clear that if the kernel of a Toeplitz operator is not reduced to  $\{0\}$  — or equivalently (since  $p = 2$ )  $g^2$  is not rigid — then it contains an *outer* function (just divide out the inner factor of any non zero function contained in the kernel). However, Toeplitz operators with non trivial kernels containing no inner functions can be easily constructed. Take for instance  $T_{\bar{z}g_0/g_0} = T_{\bar{z}}T_{g_0/g_0}$ ,

where  $g_0(z) = (1-z)^\alpha$  and  $\alpha \in (0, 1/2)$ . The Toeplitz operator  $T_{\overline{g_0}/g_0}$  is invertible ( $|g_0|^2$  satisfies the Muckenhoupt  $(A_2)$  condition) and  $(T_{\overline{g_0}/g_0})^{-1} = g_0 P_+ \frac{1}{g_0}$  [Ro77] so that the kernel of  $T_{\overline{zg_0}/g_0}$  is given by the preimage under  $T_{\overline{g_0}/g_0}$  of the constants (which define the kernel of  $T_{\overline{z}}$ ). Since  $g_0 P_+(c/\overline{g_0}) = c g_0 / \overline{g_0(0)}$ ,  $c$  being any complex number, we have  $\ker T_{\overline{zg_0}/g_0} = \mathbb{C} g_0$  which does not contain any inner function.

So, without any condition on  $g$ , we cannot hope for reasonable results. In the above example, when  $p = 2$ , then the function  $g^2(z) = (1+z)^2$  is in fact not rigid (for instance the argument of  $(1+z)^2$  is the same as that of  $z$ ). As already pointed out, rigidity of  $g^2$  is also characterized by the fact that  $T_{\overline{g}/g}$  is injective (see [Sa95, X-2]). Here  $T_{\overline{g}/g} = T_{\overline{z}}$  the kernel of which is  $\mathbb{C}$ . From this it can also be deduced that  $g^2$  is rigid if and only if  $H^p(|g|^p) \cap \overline{H^p(|g|^p)} = \{0\}$  which indicates again that rigidity should be assumed if we want to have  $K_I^p(|g|^p)$  reasonably defined. (See [Ka96] for some discussions on the intersection  $H^p(|g|^p) \cap \overline{H^p(|g|^p)}$ .)

A stronger condition than rigidity (at least when  $p = 2$ ) is that of a Muckenhoupt weight. Let us recall the Muckenhoupt  $(A_p)$  condition: for general  $1 < p < \infty$  a weight  $w$  satisfies the  $(A_p)$  condition if

$$B := \sup_{I \text{ subarc of } \mathbb{T}} \left\{ \frac{1}{|I|} \int_I w(x) dx \times \left( \frac{1}{|I|} \int_I w^{-1/(p-1)}(x) dx \right)^{p-1} \right\} < \infty.$$

When  $p = 2$ , it is known that this condition is equivalent to the so-called Helson-Szegő condition. The Muckenhoupt condition will play some rôle in the results to come. However, our main theorem on analytic continuation (Theorem 17) works under a weaker local integrability condition.

Another observation can be made now. We have already mentioned that rigidity of  $g^2$  in  $H^1$  is equivalent to injectivity of  $T_{\overline{g}/g}$ , when  $g$  is outer. It is also clear that  $T_{\overline{g}/g}$  is *always* injectif so that when  $g^2$  is rigid, the operator  $T_{\overline{g}/g}$  is injectif with dense range. On the other hand, by a result of Devinatz and Widom (see e.g. [Ni02, Theorem B4.3.1]), the invertibility of  $T_{\overline{g}/g}$ , where  $g$  is outer, is equivalent to  $|g|^2$  being  $(A_2)$ . So the difference between rigidity and  $(A_2)$  is the surjectivity (in fact the closedness of the range) of the corresponding Toeplitz operator. A criterion for surjectivity of non-injective Toeplitz operators can be found in [HSS04]. It appeals to a parametrization which was earlier used by Hayashi [Hay90] to characterize kernels of Toeplitz operators among general nearly invariant subspaces. Rigid functions do appear in the characterization of Hayashi.

As a consequence of Theorem 17 below analytic continuation can be expected on arcs not meeting the spectrum of  $I$  when  $|g|^p$  is  $(A_p)$  (see Remark 1). However the  $(A_p)$  condition cannot be expected to be necessary since it is a global condition whereas continuation depends on the local behaviour of  $I$  and  $g$ . We will even give an example of a non-rigid function  $g$  (hence not satisfying the  $(A_p)$  condition) for which analytic continuation is always possible in certain points of  $\mathbb{T}$  where  $g$  vanishes essentially.

Closely connected with the continuation problem in backward shift invariant subspaces is the spectrum of the backward shift operator on the space under consideration. The following result follows from [ARR98, Theorem 1.9]: Let  $B$  be the backward shift on  $H^p(|g|^p)$ , defined by

$Bf(z) = (f - f(0))/z$ . Clearly,  $K_I^p(|g|^p)$  is invariant with respect to  $B$  whenever  $I$  is inner. Then,  $\sigma(B|K_I^p(|g|^p)) = \sigma_{ap}(B|K_I^p(|g|^p))$ , where  $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \exists (f_n)_n \text{ with } \|f_n\| = 1 \text{ and } (\lambda - T)f_n \rightarrow 0\}$  denotes the approximate point spectrum of  $T$ , and this spectrum is equal to

$$\mathbb{T} \setminus \{1/\zeta \in \mathbb{T} : \text{every } f \in K_I^p(|g|^p) \text{ extends analytically in a neighbourhood of } \zeta\}.$$

The aim is to link this set and  $\sigma(I)$ . Here we will need the Muckenhoupt condition. Then, as in the unweighted situation, the approximate spectrum of  $B|K_I^p(|g|^p)$  on  $\mathbb{T}$  contains the conjugated spectrum of  $I$ . We will see later that the inclusion in the following proposition [H08] actually is an equality.

**Proposition 3** (Hartmann 2008). *Let  $g$  be outer in  $H^p$  such that  $|g|^p$  is a Muckenhoupt  $(A_p)$ -weight. Let  $I$  be an inner function with spectrum  $\sigma(I) = \{\lambda \in \text{clos } \mathbb{D} : \liminf_{z \rightarrow \lambda} I(z) = 0\}$ . Then  $\overline{\sigma(I)} \subset \sigma_{ap}(B|K_I^p(|g|^p))$ .*

We now come to the main result in the weighted situation (see [H08]).

**Theorem 17** (Hartmann 2008). *Let  $g$  be an outer function in  $H^p$ ,  $1 < p < \infty$  and  $I$  an inner function with associated spectrum  $\sigma(I)$ . Let  $\Gamma$  be a closed arc in  $\mathbb{T}$ . If there exists  $s > q$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , with  $1/g \in L^s(\Gamma)$ , then every function  $f \in K_I^p(|g|^p)$  extends analytically through  $\Gamma$  if and only if  $\Gamma$  does not meet  $\sigma(I)$ .*

Note that in [H08] only the sufficiency part of the above equivalence was shown. However the condition that  $\Gamma$  must not meet  $\sigma(I)$  is also necessary (even under the a priori weaker condition of continuation through  $\Gamma$ ) as follows from the proof of Theorem 18 below.

It turns also out that — like in the de Branges-Rovnyak situation discussed in Theorem 8 — for analytic continuation it is actually sufficient to have continuation. This result is new, and we will state it as a theorem provided with a proof. It is based on ideas closed to the proof of the previous theorem.

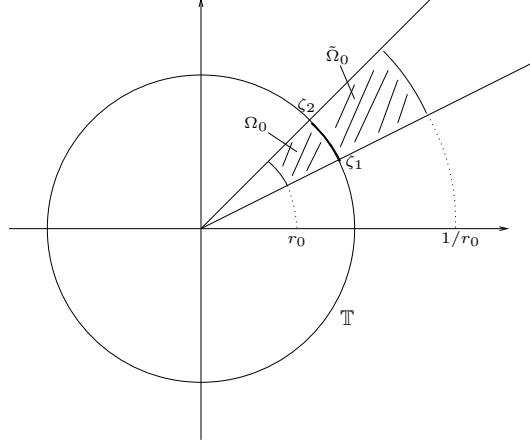
**Theorem 18.** *Let  $g$  be an outer function in  $H^p$ ,  $1 < p < \infty$  and  $I$  an inner function with associated spectrum  $\sigma(I)$ . Let  $\Gamma$  be an open arc in  $\mathbb{T}$ . Suppose that every function  $f \in K_I^p(|g|^p)$  extends continuously through  $\Gamma$  then  $\Gamma \cap \sigma(I) = \emptyset$ , and every function in  $K_I^p(|g|^p)$  extends analytically through  $\Gamma$ .*

**Proof:** Observe first that obviously  $k_\lambda^I \in K_I^2(|g|^2)$ . By the Schwarz reflection principle, in order that  $k_\lambda^I$  continues through  $\Gamma$  we need that  $\Gamma$  does not meet  $\sigma(I)$  (note that  $\text{clos } \Gamma$  could meet  $\sigma(I)$ ).

As in the unweighted situation, every meromorphic function  $f/I$ ,  $f = I\bar{\psi} \in K_I^2(|g|^2)$ , admits a pseudocontinuation  $\tilde{\psi}$ , defined by  $\tilde{\psi}(z) = \sum_{n \geq 0} \bar{\psi}(n) \frac{1}{z^n}$  in the exterior disk  $\mathbb{D}_e = \mathbb{C} \setminus \text{clos } \mathbb{D}$ .

Fix  $\Gamma_0$  any closed subarc of  $\Gamma$ . Since  $\sigma(I)$  is closed, the distance between  $\sigma(I)$  and  $\Gamma_0$  is strictly positif. Then there is a neighbourhood of  $\Gamma_0$  intersected with  $\mathbb{D}$  where  $|I(z)| \geq \delta > 0$ . It is clear that in this neighbourhood we are far away from the part of the spectrum of  $I$  contained in  $\mathbb{D}$ . Thus  $I$  extends analytically through  $\Gamma_0$ . For what follows we will call the endpoints of this arc  $\zeta_1 := e^{it_1}$  and  $\zeta_2 := e^{it_2}$  (oriented in the positive sense).

The following argument is in the spirit of Moeller [Mo62] and based on Morera's theorem. Let us introduce some notation (see Figure 1).

Figure 1: The regions  $\Omega_0$  and  $\tilde{\Omega}_0$ 

For suitable  $r_0 \in (0, 1)$  let  $\Omega_0 = \{z = re^{it} \in \mathbb{D} : t \in [t_1, t_2], r_0 \leq r < 1\}$ . and  $\tilde{\Omega}_0 = \{z = e^{it}/r \in \mathbb{D}_e : t \in [t_1, t_2], r_0 \leq r < 1\}$ . Define

$$F(z) = \begin{cases} f(z)/I(z) & \text{when } z \in \Omega_0 \\ \tilde{\psi}(z) & \text{when } z \in \tilde{\Omega}_0. \end{cases}$$

By construction this function is analytic on  $\Omega_0 \cup \tilde{\Omega}_0$  and continuous on  $\overline{\Omega_0 \cup \tilde{\Omega}_0}$ . Such a function is analytical on  $\overline{\Omega_0 \cup \tilde{\Omega}_0}$ .  $\square$

**Remark 1.** It is known (see e.g. [Mu72]) that when  $|g|^p \in (A_p)$ ,  $1 < p < \infty$ , then there exists  $r_0 \in (1, p)$  such that  $|g|^p \in (A_r)$  for every  $r > r_0$ . Take  $r \in (r_0, p)$ . Then in particular  $1/g \in L^s$ , where  $\frac{1}{r} + \frac{1}{s} = 1$ . Since  $r < p$  we have  $s > q$ . which allows to conclude that in this situation  $1/g \in L^s(\Gamma)$  for every  $\Gamma \subset \mathbb{T}$  ( $s$  independant of  $\Gamma$ ).

We promised earlier an example of a non-rigid function  $g$  for which analytic continuation of  $K_I^p$ -functions is possible in certain points where  $g$  vanishes.

*Example.* For  $\alpha \in (0, 1/2)$ , let  $g(z) = (1+z)(1-z)^\alpha$ . Clearly  $g$  is an outer function vanishing essentially in 1 and  $-1$ . Set  $h(z) = z(1-z)^{2\alpha}$ , then by similar arguments as those employed in the introducing example to this section one can check that  $\arg g^2 = \arg h$  a.e. on  $\mathbb{T}$ . Hence  $g$  is not rigid (it is the “big” zero in  $-1$  which is responsible for non-rigidity). On the other hand, the zero in  $+1$  is “small” in the sense that  $g$  satisfies the local integrability condition in a neighbourhood of 1 as required in the theorem, so that whenever  $I$  has its spectrum far from 1, then every  $K_I^2(|g|^2)$ -function can be analytically continued through suitable arcs around 1.

This example can be pushed a little bit further. In the spirit of Proposition 2 we check that (even) when the spectrum of an inner function  $I$  does not meet  $-1$ , there are functions in  $K_I^p(|g|^p)$  that are badly behaved in  $-1$ . Let again  $g_0(z) = (1-z)^\alpha$ . Then

$$\frac{\overline{g(z)}}{g(z)} = \frac{\overline{(1+z)(1-z)^\alpha}}{(1+z)(1-z)^\alpha} = \bar{z} \frac{\overline{g_0(z)}}{g_0(z)}.$$

As already explained, for every inner function  $I$ , we have  $\ker T_{\overline{I}g/g} = gK_I^p(|g|^p)$ , so that we are interested in the kernel  $\ker T_{\overline{I}g/g}$ . We have  $T_{\overline{I}g/g}f = 0$  when  $f = Iu$  and  $u \in \ker T_{\overline{g}/g} = \ker T_{\overline{g_0}/g_0} = \mathbb{C}g_0$  (see the discussion just before the proof of Proposition 2). Hence the function defined by

$$F(z) = \frac{f(z)}{g(z)} = I(z) \frac{g_0(z)}{g(z)} = \frac{I(z)}{1+z}$$

is in  $K_I^p(|g|^p)$  and it is badly behaved in  $-1$  when the spectrum of  $I$  does not meet  $-1$  (but not only).

The preceding discussions motivate the following question: does rigidity of  $g$  suffice to get analytic continuation for  $K_I^p(|g|^2)$ -function whenever  $\sigma(I)$  is far from zeros of  $g$ ?

Theorem 17 together with Proposition 3 and Remark 1 allow us to obtain the following result. We should mention that it is easy to check that  $H^p(|g|^p)$  satisfies the conditions required of a Banach space of analytic functions in order to apply the results of [ARR98].

**Corollary 2** (Hartmann 2008). *Let  $g$  be outer in  $H^p$  such that  $|g|^p$  is a Muckenhoupt  $(A_p)$  weight. Let  $I$  be an inner function with spectrum  $\sigma(I) = \{\lambda \in \text{clos } \mathbb{D} : \liminf_{z \rightarrow \lambda} I(z) = 0\}$ . Then  $\overline{\sigma(I)} = \sigma_{ap}(B|K_I^p(|g|^p))$ .*

Another simple consequence of Theorem 17 concerns embeddings. Contrarily to the situations discussed in Sections 2 and 3, the weight is here on the  $K_I^p$ -side.

**Corollary 3** (Hartmann 2008). *Let  $I$  be an inner function with spectrum  $\sigma(I)$ . If  $\Gamma \subset \mathbb{T}$  is a closed arc not meeting  $\sigma(I)$  and if  $g$  is an outer function in  $H^p$  such that  $|g| \geq \delta$  on  $\mathbb{T} \setminus \Gamma$  for some constant  $\delta > 0$  and  $1/g \in L^s(\Gamma)$ ,  $s > q$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ . Then  $K_I^p(|g|^p) \subset K_I^p$ . If moreover  $g$  is bounded, then the last inclusion is an equality.*

Suppose now  $p = 2$ . We shall use this corollary to construct an example where  $K_I^2(|g|^2) = K_I^2$  without  $g$  being extremal for  $gK_I^2(|g|^2)$ . Recall from Hitt's result [Hi88], that when  $g$  is the extremal function of a nearly invariant subspace  $M \subset H^2$ , then there exists an inner function  $I$  such that  $M = gK_I^2$ , and  $g$  is an isometric multiplier on  $K_I^2$  so that  $K_I^2 = K_I^2(|g|^2)$ . Recall from [HS03, Lemma 3] that a function  $g$  is extremal for  $gK_I^2(|g|^2)$  if  $\int f|g|^2 dm = f(0)$  for every function  $f \in K_I^2(|g|^2)$ . Our example is constructed in the spirit of [HS03, p.356]. Fix  $\alpha \in (0, 1/2)$ . Let  $\gamma(z) = (1 - z)^\alpha$  and let  $g$  be an outer function in  $H^2$  such that  $|g|^2 = \text{Re } \gamma$  a.e. on  $\mathbb{T}$  (such a function clearly exists). Let now  $I = B_\Lambda$  be an infinite Blaschke product with  $0 \in \Lambda$ . If  $\Lambda$  accumulates to points outside 1, then the corollary shows that  $K_I^2 = K_I^2(|g|^2)$ . Let us check that  $g$  is not extremal. To this end we compute  $\int k_\lambda |g|^2 dm$  for  $\lambda \in \Lambda$  (recall that for  $\lambda \in \Lambda$ ,  $k_\lambda \in K_I^2 = K_I^2(|g|^2)$ ):

$$\begin{aligned} \int k_\lambda |g|^2 dm &= \int k_\lambda \text{Re } \gamma dm = \frac{1}{2} \left( \int k_\lambda \gamma dm + \int k_\lambda \overline{\gamma} dm \right) = \frac{1}{2} k_\lambda(0) \gamma(0) + \frac{1}{2} \langle k_\lambda, \gamma \rangle \\ (26) \quad &= \frac{1}{2} (1 + \overline{(1 - \lambda)^\alpha}) \end{aligned}$$

which is different from  $k_\lambda(0) = 1$  (except when  $\lambda = 0$ ). Hence  $g$  is not extremal.



We could also have obtained the non-extremality of  $g$  from Sarason's result [Sa89, Theorem 2] using the parametrization  $g = \frac{a}{1-b}$  appearing in Sarason's and Hayashi's work (see [H08] for details on this second argument).

It is clear that the corollary is still valid when  $\Gamma$  is replaced by a finite union of intervals. However, we can construct an infinite union of intervals  $\Gamma = \bigcup_{n \geq 1} \Gamma_n$  each of which does not meet  $\sigma(I)$ , an outer function  $g$  satisfying the yet weaker integrability condition  $1/g \in L^s(\Gamma)$ ,  $s < 2$ , and  $|g| \geq \delta$  on  $\mathbb{T} \setminus \Gamma$ , and an inner function  $I$  such that  $K_I^2(|g|^2) \not\subset K_I^2$ . The function  $g$  obtained in this construction does not satisfy  $|g|^2 \in (A_2)$ . (See [H08] for details.)

Another simple observation concerning the local integrability condition  $1/g \in L^s(\Gamma)$ ,  $s > q$ : if it is replaced by the global condition  $1/g \in L^s(\mathbb{T})$ , then by Hölder's inequality we have an embedding into a bigger backward shift invariant subspace:

**Proposition 4** (Hartmann 2008). *Let  $1 < p < \infty$  and  $1/p + 1/q = 1$ . If there exists  $s > q$  such that  $1/g \in L^s(\mathbb{T})$ , then for  $r$  with  $1/r = 1/p + 1/s$  we have  $L^p(|g|^p) \subset L^r$ .*

So in this situation we of course also have  $K_I^p(|g|^p) \subset K_I^r$ . In particular, every function  $f \in K_I^p(|g|^p)$  admits a pseudocontinuation and extends analytically outside  $\sigma(I)$ . Again the Ahern-Clark condition does not give complete information for the points located in the spectrum of  $I$  since (ii) of Theorem 2 depends on  $p$ .

When one allows  $g$  to vanish in points contained in  $\sigma(I)$ , then it is possible to construct examples with  $|g|^p \in (A_p)$  and  $K_I^p(|g|^p) \not\subset K_I^p$ : take for instance  $I = B_\Lambda$  the Blasche product vanishing exactly in  $\Lambda = \{1 - \frac{1}{2^n}\}_n$  and  $g(z) = (1 - z)^\alpha$ , where  $\alpha \in (0, 1/2)$  and  $p = 2$  (see [H08] for details; the condition  $|g|^2 \in (A_2)$  is required in the proof to show that  $K_I^2(|g|^2) = P_+(\frac{1}{g}K_I^2)$  — see Lemma 2 below — which gives an explicit description of  $K_I^2$  in terms of coefficients with respect to an unconditional basis). The following crucial example is in the spirit of this observation.

**4.1. An example.** In the spirit of the example given in [H08, Proposition 4] we shall now discuss the condition (ii) of Theorem 2 in the context of weighted backward shift invariant subspaces.

We first have to recall Lemma 1 from [H08]:

**Lemma 2** (Hartmann 2008). *Suppose  $|g|^p$  is an  $(A_p)$  weight and  $I$  an inner function. Then  $A_0 = P_+ \frac{1}{g}$  is an isomorphism of  $K_I^p$  onto  $K_I^p(|g|^p)$ . Also, for every  $\lambda \in \mathbb{D}$  we have*

$$(27) \quad A_0 k_\lambda = \frac{k_\lambda(\mu)}{g(\lambda)}.$$

We return to the situation  $p = 2$ . Take  $g(z) = (1 - z)^\alpha$  with  $\alpha \in (0, 1/2)$ . Then  $|g|^2$  is  $(A_2)$ . Let

$$r_n = 1 - \frac{1}{2^n}, \quad \theta_n = (1 - r_n)^s = \frac{1}{2^{ns}}, \quad \lambda_n = r_n e^{i\theta_n},$$

where  $s \in (0, 1/2)$ . Hence the sequence  $\Lambda = \{\lambda_n\}_n$  tends tangentially to 1. Set  $I = B_\Lambda$ . We check the Ahern-Clark condition in  $\zeta = 1$  for  $N = 0$  (which means that we are just interested in

the existence of non tangential limits in  $\zeta = 1$ ). Observe that for  $s \in (0, 1/2)$  we have

$$(28) \quad |1 - r_n e^{i\theta_n}|^2 \simeq (1 - r_n)^2 + \theta_n^2 = \frac{1}{2^{2n}} + \frac{1}{2^{2ns}} \simeq \frac{1}{2^{2ns}},$$

and so when  $q > 1$

$$(29) \quad \sum_{n \geq 1} \frac{1 - r_n^2}{|1 - r_n e^{i\theta_n}|^q} \simeq \sum_{n \geq 1} \frac{1/2^n}{1/2^{nqs}} \simeq \sum_{n \geq 1} 2^{n(sq-1)}.$$

The latter sum is bounded when  $q = 2$  which implies in the unweighted situation that every function in the backward shift invariant subspace  $K_I^2$  has a nontangential limit at 1. Note also that since  $|g|^2 \in (A_2)$ , by Proposition 4 and comments thereafter,  $K_I^2(|g|^2)$  imbeds into some  $K_I^r$ ,  $r < 2$ . Now taking  $q = r' > 2$ , where  $\frac{1}{r} + \frac{1}{r'} = 1$ , we see that the sum in (29) diverges when  $sr' \geq 1$  and converges for  $rs' < 1$ . So depending on the parameters  $s$  and  $\alpha$  we can assert continuation or not. It will be clear a posteriori that the parameter  $s$  — ensuring convergence of (29) for  $q = 2$  — will be such that  $r's \geq 1$  ( $r$  and hence  $r'$  are fixed once  $\alpha$  is fixed).

Note that  $\sigma(I) \cap \mathbb{T} = \{1\}$ , which corresponds to the point where  $g$  vanishes. Clearly,  $\Lambda$  is an interpolating sequence, and so the sequence  $\{k_{\lambda_n}/\|k_{\lambda_n}\|_2\}_n$  is a normalized unconditional basis in  $K_I^2$ . This means that we can write  $K_I^2 = l^2(\frac{k_{\lambda_n}}{\|k_{\lambda_n}\|_2})$  meaning that  $f \in K_I^2$  if and only if

$$f = \sum_{n \geq 1} \alpha_n \frac{k_{\lambda_n}}{\|k_{\lambda_n}\|_2}$$

with  $\sum_{n \geq 1} |\alpha_n|^2 < \infty$  (the last sum defines the square of an equivalent norm in  $K_I^2$ ).

As already mentioned  $|g|^2$  is Muckenhoupt  $(A_2)$ . This implies in particular that we have the local integrability condition  $1/g \in L^s(\Gamma)$  for some  $s > 2$  and  $\Gamma$  an arc containing the point 1. Moreover, we get from (27)

$$\{A_0(k_{\lambda_n}/\|k_{\lambda_n}\|_2)\}_n = \left\{ \frac{k_{\lambda_n}}{g(\lambda_n)\|k_{\lambda_n}\|_2} \right\}_n,$$

and  $\{k_{\lambda_n}/(\overline{g(\lambda_n)}\|k_{\lambda_n}\|_2)\}_n$  is an unconditional basis in  $K_I^2(|g|^2)$  (almost normalized in the sense that  $\|A_0(k_{\lambda_n}/\|k_{\lambda_n}\|_2)\|_{|g|^2}$  is comparable to a constant independent of  $n$ ). Hence for every sequence  $\alpha = (\alpha_n)_n$  with  $\sum_{n \geq 1} |\alpha_n|^2 < \infty$ , we have

$$f_\alpha := \sum_{n \geq 1} \frac{\alpha_n}{g(\lambda_n)} \frac{k_{\lambda_n}}{\|k_{\lambda_n}\|_2} \in K_I^2(|g|^2).$$

To fix the ideas we will now pick  $\alpha_n = 1/n^{1/2+\varepsilon}$  for some  $\varepsilon > 0$  so that  $\sum_n \alpha_n k_{\lambda_n}/\|k_{\lambda_n}\|_2$  is in  $K_I^2$ , and hence  $f_\alpha \in K_I^2(|g|^2)$ . Let us show that  $f$  does not have a non tangential limit in 1. Fix  $t \in (0, 1)$ . Then

$$f(t) = \sum_n \frac{\alpha_n}{g(\lambda_n)} \frac{k_{\lambda_n}(t)}{\|k_{\lambda_n}\|_2}.$$

We have  $\|k_{\lambda_n}\|_2 = 1/\sqrt{1 - |\lambda_n|^2} \simeq 2^{n/2}$ . Also as in (28),

$$|g(\lambda_n)| = |1 - \lambda_n|^\alpha \simeq \theta_n^\alpha = \frac{1}{2^{n\alpha}}$$

Changing the arguments of the  $\alpha_n$ 's and renormalizing, we can suppose that

$$\frac{\alpha_n}{g(\lambda_n)\|k_{\lambda_n}\|_2} = \frac{2^{n(s\alpha-1/2)}}{n^{1/2+\varepsilon}}$$

Let us compute the imaginary part of  $f_\alpha$  in  $t$ . Observe that the imaginary part of  $1/(1 - t\bar{\lambda}_n)$  is negative. More precisely, assuming  $t \in [1/2, 1)$  and  $n \geq N_0$ ,

$$\operatorname{Im} \frac{1}{1 - t\bar{\lambda}_n} = \operatorname{Im} \frac{1 - t\lambda_n}{|1 - t\bar{\lambda}_n|^2} = \frac{-tr_n \sin \theta_n}{|1 - t\bar{\lambda}_n|^2} \simeq \frac{-\theta_n}{|1 - t\bar{\lambda}_n|^2} = \frac{-1/2^{ns}}{|1 - t\bar{\lambda}_n|^2}$$

Also for  $n \geq N = \log_2(1/(1-t))$ , we have  $1-t \geq 1/2^n$  and  $r_n = 1 - 1/2^n \geq t$ , so that for these  $n$

$$\begin{aligned} |1 - t\bar{\lambda}_n|^2 &\simeq (1 - tr_n)^2 + \theta_n^2 \leq (1 - t^2)^2 + \theta_n^2 \leq 4(1-t)^2 + \theta_N^2 \\ &\leq 4(1-t)^2 + c(1-t)^{2s} \lesssim (1-t)^{2s} \end{aligned}$$

So

$$\begin{aligned} |\operatorname{Im} f(t)| &= \left| \operatorname{Im} \sum_n \frac{\alpha_n}{g(\lambda_n)} \frac{k_{\lambda_n}(t)}{\|k_{\lambda_n}\|_2} \right| \gtrsim \sum_{n \geq \log_2(1/(1-t))} \frac{2^{n(s\alpha-1/2)}}{n^{1/2+\varepsilon}} \frac{1/2^{ns}}{(1-t)^{2s}} \\ &\gtrsim \frac{1}{(1-t)^{2s}} \sum_{n \geq \log_2(1/(1-t))} \frac{1}{2^{\gamma n}} \simeq \frac{1}{(1-t)^{2s}} \frac{1}{2^{\gamma \log_2(1/(1-t))}} \\ &\gtrsim (1-t)^{\gamma-2s}, \end{aligned}$$

where  $\gamma = s + 1/2 - s\alpha + \delta$  for an arbitrarily small  $\delta$  (this compensates the term  $n^{1/2+\varepsilon}$ ). So  $\gamma - 2s = 1/2 - s(1+\alpha) + \delta$  which can be made negative by choosing  $s$  closely enough to  $1/2$ .

We conclude that the function  $f_\alpha$  is not bounded in 1 and thus cannot have a non-tangential limit in  $\zeta = 1$

## REFERENCES

- [AC69] P.R. Ahern, D.N. Clark, *Invariant subspaces and analytic continuation in several variables*, J. Math. Mech. **19** (1969/1970), 963–969.
- [AC70] P.R. Ahern, D.N. Clark, *Radial limits and invariant subspaces*, Amer. J. Math. **92** (1970), 332–342.
- [AI99] A.B. Aleksandrov, *Embedding theorems for coinvariant subspaces of the shift operator. II.* (Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **262** (1999), 5–48; English translation in J. Math. Sci. (New York) **110** (2002), 2907–2929.
- [AI95] A.B. Aleksandrov, *On the existence of angular(nontangential) boundary values of pseudocontinuable functions.* (Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **222** (1995), Issled. po Linein. Oper. i Teor. Funktsii. 23, 5–17, 307; translation in J. Math. Sci. (New York) **87** (1997), no. 5, 3781–3787.

- [Al89] A.B. Aleksandrov, *Inner functions and related spaces of pseudocontinuable functions*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **170** (1989), Issled. Linein. Oper. Teorii Funktsii. 17, 7–33, 321; translation in J. Soviet Math. **63** (1993), no. 2, 115–129
- [Al81] A.B. Aleksandrov, *Invariant subspaces of shift operators. An axiomatic approach*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **113** (1981), 7–26; English transl.: J. Soviet Math **22** (1983), 115–129.
- [ARR98] A. Aleman, S. Richter, W. Ross, *Pseudocontinuations and the backward shift*. Indiana Univ. Math. J. **47** (1998), no. 1, 223–276.
- [AR06] J.M. Anderson, J. Rovnyak, *On generalized Schwarz–Pick estimates*, Mathematika **53** (2006), no. 1, 161–168.
- [Ba03] A.D. Baranov, *Weighted Bernstein-type inequalities and embedding theorems for shift-coinvariant subspaces*, Algebra i Analiz **15** (2003), no. 5, 138–168; English transl.: St. Petersburg Math. J. **15** (2004), no. 5, 733–752.
- [Ba05a] A.D. Baranov, *Bernstein-type inequalities for the shift-coinvariant subspaces and their applications to Carleson embeddings*, J. Funct. Anal. **223** (2005), 116–146.
- [Ba05b] A.D. Baranov, *Stability of bases and frames of reproducing kernels in model subspaces*, Ann. Inst. Fourier (Grenoble) **55** (2005), 2399–2422.
- [Ba08] A.D. Baranov, *Compact and Schatten class embeddings of star-invariant subspaces in the disc*, to appear in Izv. Math.
- [BFM09] A.D. Baranov, E. Fricain and J. Mashregi, *Weighted norm inequalities for de Branges–Rovnyak spaces and their applications*, to appear in Amer. J. Math.
- [Bern26] S.N. Bernstein, *Leçons sur les propriétés extrémales et la meilleurs approximation des fonctions analytiques d’une variable réelle*, Gauthier-Villars, Paris, 1926.
- [Be72] A. Beurling, *Analytic continuation across a linear boundary*, Acta Math. **128** (1972), no. 3–4, 153–182.
- [BK06] V. Bolotnikov, A. Kheifets, *A higher order analogue of the Carathéodory–Julia theorem*. J. Funct. Anal. **237** (2006), 350–371.
- [Ca58] L. Carleson, *An interpolation problem for bounded analytic functions*, Amer. J. Math. **80** (1958), 921–930.
- [Ca62] L. Carleson, *Interpolations by bounded analytic functions and the corona problem*, Ann. of Math. (2) **76** (1962), 547–559.
- [CM03] J.A. Cima, A.L. Matheson, *On Carleson embeddings of star-invariant subspaces*, Quaest. Math. **26** (2003), 3, 279–288.
- [CR00] J. A. Cima & W. T. Ross, *The Backward Shift on the Hardy Space*, Math. Surveys Monographs **79**, Amer. Math. Soc., Providence R.I., 2000.
- [Cl72] D.N. Clark, *One dimensional perturbations of restricted shifts*. J. Analyse Math. **25** (1972), 169–191.
- [Co82] W.S. Cohn, *Carleson measures for functions orthogonal to invariant subspaces*, Pacific J. Math. **103** (1982), 347–364.
- [Co86a] W. S. Cohn, *Radial limits and star invariant subspaces of bounded mean oscillation*, Amer. J. Math. **108** (1986), 719–749.
- [Co86b] W.S. Cohn, *Carleson measures and operators on star-invariant subspaces*, J. Oper. Theory, **15** (1986), no. 1, 181–202.
- [dBR66a] L. de Branges, J. Rovnyak, *Canonical models in quantum scattering theory*, pp. 295–392 in: Perturbation theory and its application in quantum mechanics, Madison, 1965, ed. C.H. Wilcox, Wiley, N.Y., 1966.
- [dBR66b] L. de Branges, J. Rovnyak, *Square summable power series*, Holt, Rinehart and Winston, N.Y., 1966.
- [DSS70] R.G. Douglas, H.S. Shapiro & A.L. Shields, *Cyclic vectors and invariant subspaces for the backward shift operator*, Ann. Inst. Fourier (Grenoble) **20** 1970 fasc. 1, 37–76.
- [Du70] P.L. Duren, *Theory of  $H^p$  spaces*, Academic Press, 1970.
- [Dy08b] K.M. Dyakonov, *Toeplitz operators and arguments of analytic functions*, to appear in Math. Ann.

- [Dy08a] K.M. Dyakonov, *Meromorphic functions and their derivatives: equivalence of norms*, Indiana Univ. Math. J. **57** (2008), no. 4, 1557–1571.
- [Dy07] K.M. Dyakonov, *Two theorems on star-invariant subspaces of BMOA*, Indiana Univ. Math. J. **56** (2007), no. 2, 643–658.
- [Dy02a] K.M. Dyakonov, *Differentiation in star-invariant subspaces II: Schatten class criteria*, J. Funct. Anal. **192** (2002), 387–409.
- [Dy02b] K.M. Dyakonov, *Differentiation in star-invariant subspaces I: boudedness and compactness*, J. Funct. Anal. **192** (2002), 364–386.
- [Dy00] K.M. Dyakonov, *Continuous and compact embeddings between star-invariant subspaces*, in Oper. Theory Adv. Appl., Complex analysis, operators, and related topics **113** (2000), 65–76.
- [Dy96] K.M. Dyakonov, *Kernels of Toeplitz operators, smooth functions and Bernstein-type inequalities*. (Russian. English summary) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **201** (1992), Issled. po Linein. Oper. Teor. Funktsii. **20**, 5–21, 190; translation in J. Math. Sci. **78** (1996), no. 2, 131–141
- [Dy91] K.M. Dyakonov, *Entire functions of exponential type and model subspaces in  $H^p$* , Zap. Nauchn. Sem. Leningrad Otdel. Mat. Inst. Steklov. (LOMI) **190** (1991), 81–100; Engl. transl.: J. Math. Sci. **71** (1994), 2222–2233.
- [DK06] K.M. Dyakonov & D. Khavinson, *Smooth functions in star-invariant subspaces*, in Recent advances in operator-related function theory, Contemp. Math., **393**, Amer. Math. Soc. (2006), 59–66.
- [Fr05] E. Fricain, *Bases of reproducing kernels in de Branges spaces*, J. Funct. Anal. **226**, no 2 (2005), 373–405.
- [FM08a] E. Fricain, J. Mashregi, *Boundary behavior of functions of the de Branges-Rovnyak spaces*, Complex Anal. Oper. Theory, **2** (2008), 87–97.
- [FM08b] E. Fricain, J. Mashregi, *Integral representation of the  $n$ -th derivative in de Branges-Rovnyak spaces and the norm convergence of its reproducing kernel*, Annales de l’Institut Fourier, to appear.
- [H08] A. Hartmann, *Analytic continuations and embeddings*, preprint.
- [HSS04] A. Hartmann, D. Sarason & K. Seip, *Surjective Toeplitz operators*, Acta Sci. Math. (Szeged) **70** (2004), no. 3-4, 609–621.
- [HS03] A. Hartmann & K. Seip, *Extremal functions as divisors for kernels of Toeplitz operators*, J. Funct. Anal. **202** (2003), no. 2, 342–362.
- [Hay90] E. Hayashi, *Classification of nearly invariant subspaces of the backward shift*, Proc. Amer. Math. Soc. **110** (1990), 441–448.
- [Hi88] D. Hitt, *Invariant subspaces of  $H^2$  of an annulus*, Pacific J. Math. **134** (1988), no. 1, 101–120.
- [HNP81] S.V. Hruščëv, N.K. Nikolskii & B.S. Pavlov, *Unconditional bases of exponentials and of reproducing kernels*, Complex analysis and spectral theory (Leningrad, 1979/1980), pp. 214–335, Lecture Notes in Math., **864**, Springer, Berlin-New York, 1981.
- [Ju07] M.T. Jury, *Reproducing kernels, de Branges-Rovnyak spaces, and norms of weighted composition operators*, Proc. Amer. Math. Soc., **135** (2007), no. 11, 3669–3675.
- [Ka96] V.V. Kapustin, *Real functions in weighted Hardy spaces*, (Russian. English, Russian summary) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **262** (1999), Issled. po Linein. Oper. i Teor. Funkts. **27**, 138–146, 233–234; translation in J. Math. Sci. (New York) **110** (2002), no. 5, 2986–2990.
- [Le74] M.B. Levin, *An estimate of the derivative of a meromorphic function on the boundary of domain*, Soviet Math. Dokl. **15** (1974), 831–834.
- [Le96] B.Ya. Levin, *Lectures on entire functions*, (English summary). In collaboration with and with a preface by Yu. Lyubarskii, M. Sodin and V. Tkachenko. Translated from the Russian manuscript by Tkachenko. Translations of Mathematical Monographs, 150. American Mathematical Society, Providence, RI, 1996. xvi+248 pp. ISBN: 0-8218-0282-8

- [Mo62] J.W. Moeller, *On the spectra of some translation invariant spaces*. J. Math. Anal. Appl. **4** (1962) 276–296.
- [Mu72] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.
- [NV02] F. Nazarov, A. Volberg, *The Bellman function, the two-weight Hilbert transform, and embeddings of the model space  $K_\Theta$* , J. Anal. Math. **87** (2002), 385–414.
- [Ni02] N. Nikolski, *Operators, Functions, and Systems: An Easy Reading. Vol. 1, Hardy, Hankel, and Toeplitz; Vol.2, Model Operators and Systems*, Math. Surveys Monographs **92** and **93**, Amer. Math. Soc., Providence, RI, 2002.
- [PS06] A. Poltoratski & D. Sarason, *Aleksandrov-Clark measures*. Recent advances in operator-related function theory, 1–14, Contemp. Math., 393, Amer. Math. Soc., Providence, RI, 2006.
- [Ro77] R. Rochberg, *Toeplitz operators on weighted  $H^p$  spaces*. Indiana Univ. Math. J. **26** (1977), no. 2, 291–298.
- [Sa86] D. Sarason, *Shift-invariant spaces from the Brangesian point of view*, In *The Bieberbach Conjecture- Proceedings of the Symposium on the Occasion of the Proof*. Amer. Math. Soc., Providence, 1986, pp. 153–166.
- [Sa89] D. Sarason, *Nearly invariant subspaces of the backward shift*, 481–493, Oper. Theory Adv. Appl., 35, Birkhuser, Basel, 1988.
- [Sa89] D. Sarason, *Exposed points in  $H^1$ , I*, The Gohberg anniversary collection, Vol. II (Calgary, AB, 1988), 485–496, Oper. Theory Adv. Appl., 41, Birkhuser, Basel, 1989.
- [Sa94] D. Sarason, *Kernels of Toeplitz operators*, Toeplitz operators and related topics (Santa Cruz, CA, 1992), 153–164, Oper. Theory Adv. Appl., **71**, Birkhuser, Basel, 1994.
- [Sa95] D. Sarason, *Sub-Hardy Hilbert spaces in the unit disk*, University of Arkansas Lecture Notes in the Mathematical Sciences, 10. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1994. xvi+95 pp. ISBN: 0-471-04897-6.
- [Sh67] H.S. Shapiro, *Some observations concerning weighted polynomial approximation of holomorphic functions*, (Russian) Mat. Sb. (N.S.) **73** (115) 1967 320–330.
- [Sh01] J.E. Shapiro, *Relative angular derivatives*, J. Operator Theory **46** (2001), no. 2, 265–280.
- [Sh03] J.E. Shapiro, *More relative angular derivatives*, J. Operator Theory **49** (2003), no. 1, 85–97.
- [SNF67] Béla Sz.-Nagy & Ciprian Foias, *Analyse harmonique des opérateurs de l’espace de Hilbert*, Masson et Cie, Paris, 1967.
- [TV96] S.R. Treil & A.L. Volberg, *Weighted embeddings and weighted norm inequalities for the Hilbert transform and the maximal operator*. Algebra i Analiz **7** (1995), no. 6, 205–226; translation in St. Petersburg Math. J. **7** (1996), no. 6, 1017–1032.
- [Vo81] A.L. Volberg, *Thin and thick families of rational fractions*, Lect. Notes in Math. **864** (1981), 440–481.